Electronics

ELECTRICITY & ELECTRO-NICS

D.C. TAYAL

1

Vectors

1.1. SCALARS AND VECTORS

The physical quantities which are completely defined by magnitude alone, are called *scalars*. They are represented by *Latin* or *Greek* letters. They obey the ordinary laws of algebra. Examples are mass, charge, time, temperature, etc. In addition, there are other physical quantities which are completely determined by a magnitude and a direction in space and follow the parallelogram law of addition. These are called *vectors* and are represented by *bold face letters* or *Greek letters with arrow over them*. They are represented geometrically by an arrow pointing in the direction of the vector and of length equal to its magnitude. Examples are velocity, displacement, force, momentum, electric and magnetic fields, etc. The vectors may be classified as :(1) polar vector and (2) axial vector. For the quantity, in which more linear action in a particular direction is involved, the vector is called a *polar vector* (*e.g.*, displacement, velocity, force, etc). For the quantity, in which rotary action of some kind takes place about an axis, the vector is called *axial vector* (e.g., angular velocity, angular acceleration, etc.).

A vector which has no magnitude is called a *null* or *zero* vector. If it is added with any vector **A**, the vector **A** will always remain unchanged, *i.e.*,

$$\mathbf{0}+\mathbf{A}=\mathbf{A}. \qquad \dots (1)$$

A vector whose module is unity is called *unit vector*. It is represented by a letter with a cap over it. On multiplying a scalar with the unit vector we get a vector in the direction of the unit vector, *i.e.*,

$$\mathbf{A}=A\hat{A}. \qquad \dots (2)$$

Two vectors **A** and **B** are said to be equal if and only if they have the same magnitude and the same direction, *i.e.*,

$$\mathbf{A}=\mathbf{B} \text{ or } A\hat{A}=B\hat{B}, \text{ if } A=B \text{ and } \hat{A}=\hat{B}. \qquad \dots (3)$$

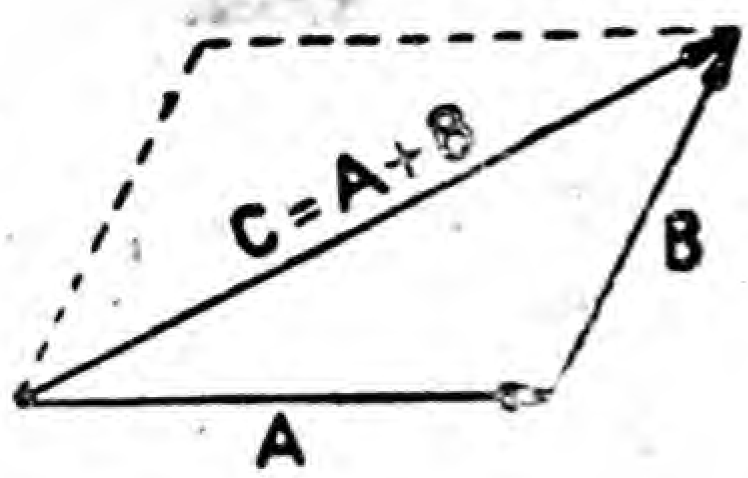


Fig. 1.1

The *vector sum* of two vectors can be obtained as follows : Draw the vector **A** and **B**, as in Fig. 1.1, at any point, with the initial point of **B** (tail of **B**) at the end point of **A** (head of **A**). The vector sum **C** is the vector from the initial point of **A** upto the end point of **B**. We thus have

$$\mathbf{C}=\mathbf{A}+\mathbf{B}. \qquad \dots (4)$$

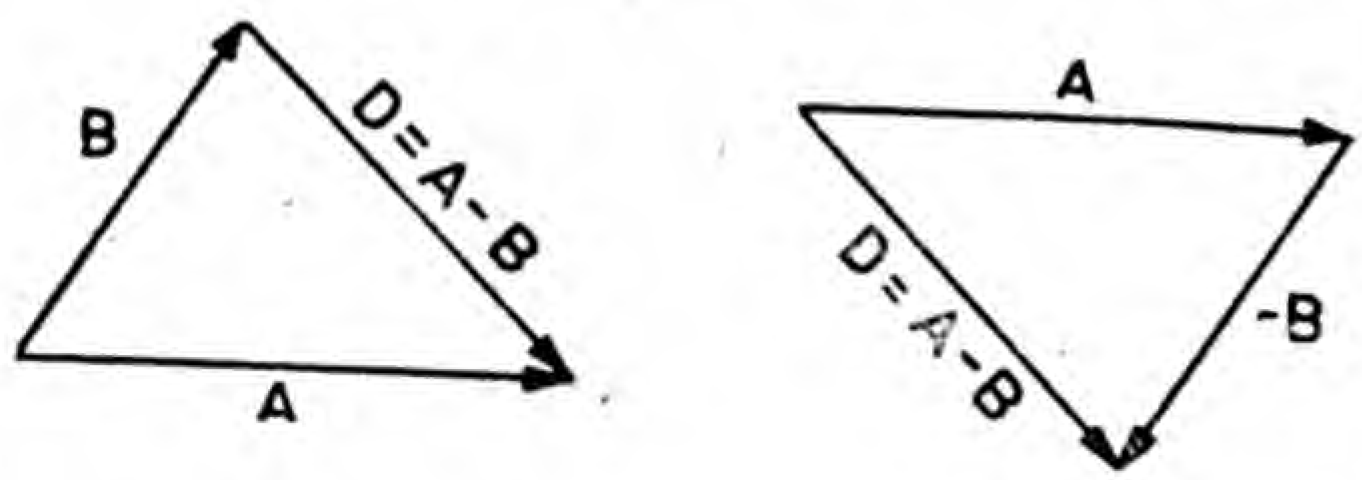
C is the diagonal of the parallelogram with sides **A** and **B** and its magnitude and direction can be found by trigonometry (Fig 1.1)

The resultant **C** may also be obtained by adding vector **B** with vector **A** (*i.e.*, **B**+**A**=**C**). Thus the *vector addition is commutative.* This method may be extended for addition of three or more vectors. The summation is independent of the order and grouping of terms, *e.g.*,

$$\mathbf{A}+\mathbf{B}+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}. \qquad \dots (5)$$

Hence the *vector sum is associative.*

For *vector subtraction,* vectors **A** and **B** are drawn from the



(*a*) (*b*)

Fig 1.2

common origin and the difference **A**−**B** is the vector from the head of **B** to head of **A** [Fig 1.2 (*a*)]. Thus

$$\mathbf{D}=\mathbf{A}-\mathbf{B}. \qquad \dots (6)$$

As the vector −**B** is the vector of magnitude *B* but possesses an opposite direction hence the difference **A**−**B** may be obtained by adding to the vector **A** a vector −**B** [Fig 1.2 (b)].

If **A** and **B** are parallel vectors, then the magnitude of **C** is equal to the scalar sum of the magnitudes of **A** and **B** and the magnitude of **D** is equal to the scalar difference.

The *converse process* is often useful, a vector **C** may be resolved into two vectors **A** and **B**, such that **C** is the diagonal of a parallelogram of the adjacent side **A** and **B**. Generally **A** and **B** are chosen to be at right angles. Any vector in space may be solved into three components parallel to the axes of cartesian dinates.

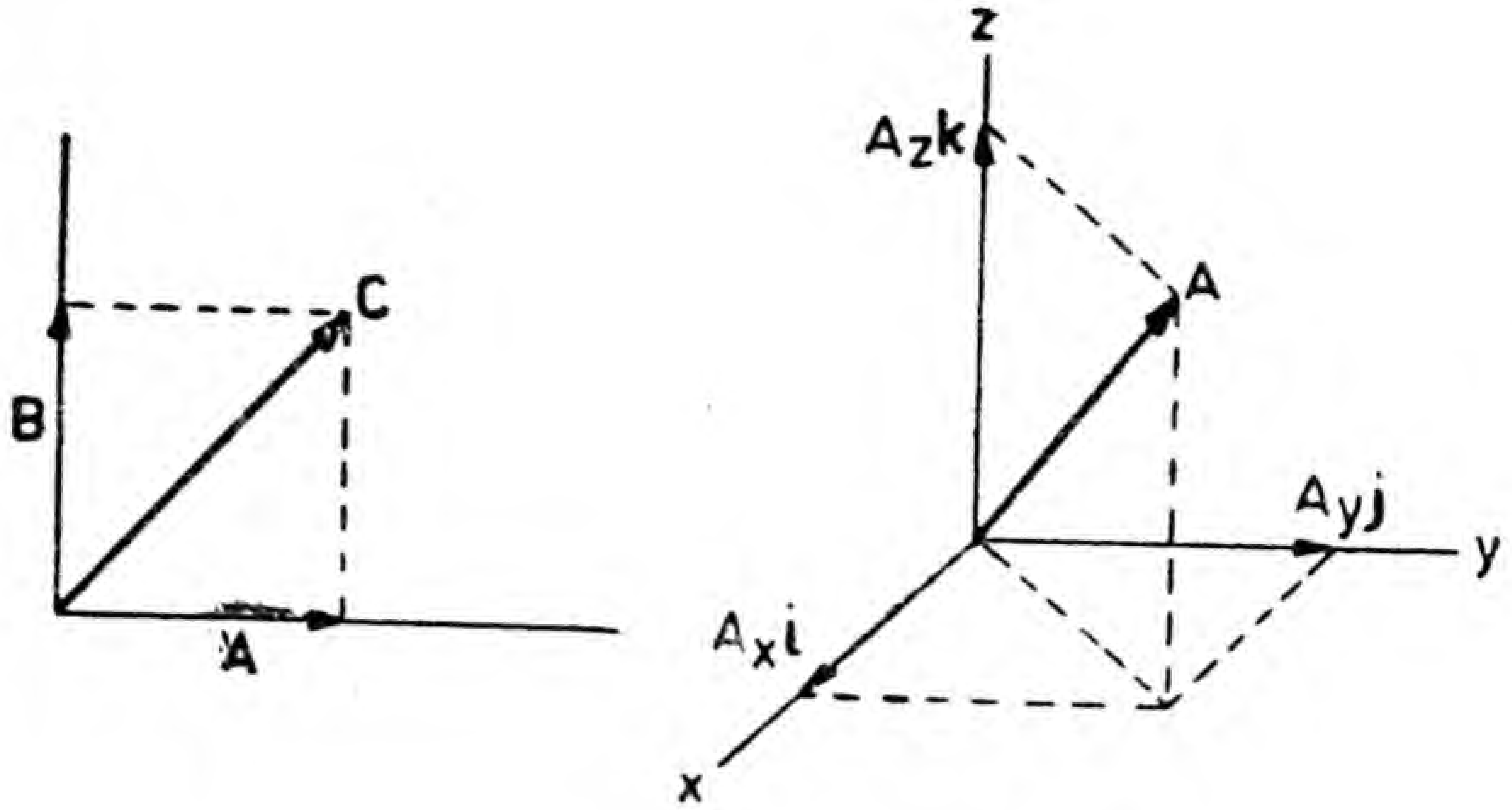


Fig. 1.3

If A_x, A_y and A_z are the magnitudes of components of the vector **A** parallel to the axes x, y and z respectively, then

$$\mathbf{A}=A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}. \quad \ldots(7)$$

where **i**, **j** and **k** are vectors of unit length along the axes x, y and z respectively. The sum of two vectors may thus be written as

$$\mathbf{A}+\mathbf{B}=(A_x+B_x)\mathbf{i}+(A_y+B_y)\mathbf{j}+(A_z+B_z)\mathbf{k}. \quad \ldots(8)$$

1.2 MULTIPLICATION OF VECTORS

When a vector **A** is multiplied by a scalar b, a new vector **C** is formed whose direction is the same as that of the original vector **A** and whose magnitude is the product of the scalar b and the magnitude A of the original vector **A**. Thus

$$\mathbf{C}=b\mathbf{A}=b[A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}]. \quad \ldots(9)$$

Similarly

$$(a+b)\,\mathbf{A}=a\mathbf{A}+b\mathbf{A}. \quad \ldots(10)$$

Following two types of vector multiplication have been defined.

(*i*) **Scalar Product**. The scalar product of two vectors **A** and **B** is a scalar quantity. It is indicated by a dot, placed between the two vectors and thus is also known as *dot-product*. Its magnitude is equal to the product of the magnitudes of two vectors and the cosine of the angle between them.

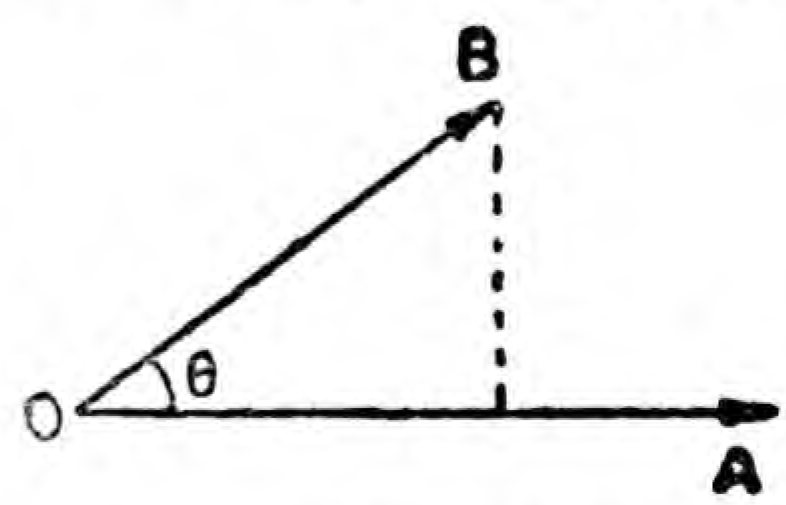


Fig. 1.4

$$\mathbf{A}.\mathbf{B}=AB\cos\theta=\mathbf{B}.\mathbf{A} \quad \ldots(11)$$

This relation shows that the scalar product is commutative. As $B\cos\theta$ = projection of **B** along **A**, hence the scalar product may be defined as *the scalar quantity numerically equal to the magnitude of one vector multiplied by the component of the other along the first.*

The scalar product of two perpendicular vectors is zero. Thus for the unit vectors **i**, **j** and **k**, we have

$$\mathbf{i}.\mathbf{j}=\mathbf{j}.\mathbf{k}=\mathbf{k}.\mathbf{i}=0 \quad \text{and} \quad \mathbf{i}.\mathbf{i}=\mathbf{j}.\mathbf{j}=\mathbf{k}.\mathbf{k}=1. \qquad ...(12)$$

$$\therefore \quad \mathbf{A}.\mathbf{B}=(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}).(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k})$$

$$=A_xB_x+A_yB_y+A_zB_z. \qquad ...(13)$$

Relation (11) shows that

$\mathbf{A}.\mathbf{B}=AB\cos 0=AB,$ if $\mathbf{A} \parallel \mathbf{B}$

$\mathbf{A}.\mathbf{B}=AB\cos\pi=-AB$ if $\theta=\pi$

$\mathbf{A}.\mathbf{B}=0$ if either $\theta=\pi/2$ or both or at least one vector is zero.

(*ii*) **Vector Product.** The *vector product of two vectors is a vector quantity.* It is indicated by placing a cross between the vectors, hence is also called *cross product.* Its magnitude is equal to the product of the magnitudes of two vectors and the sine of the

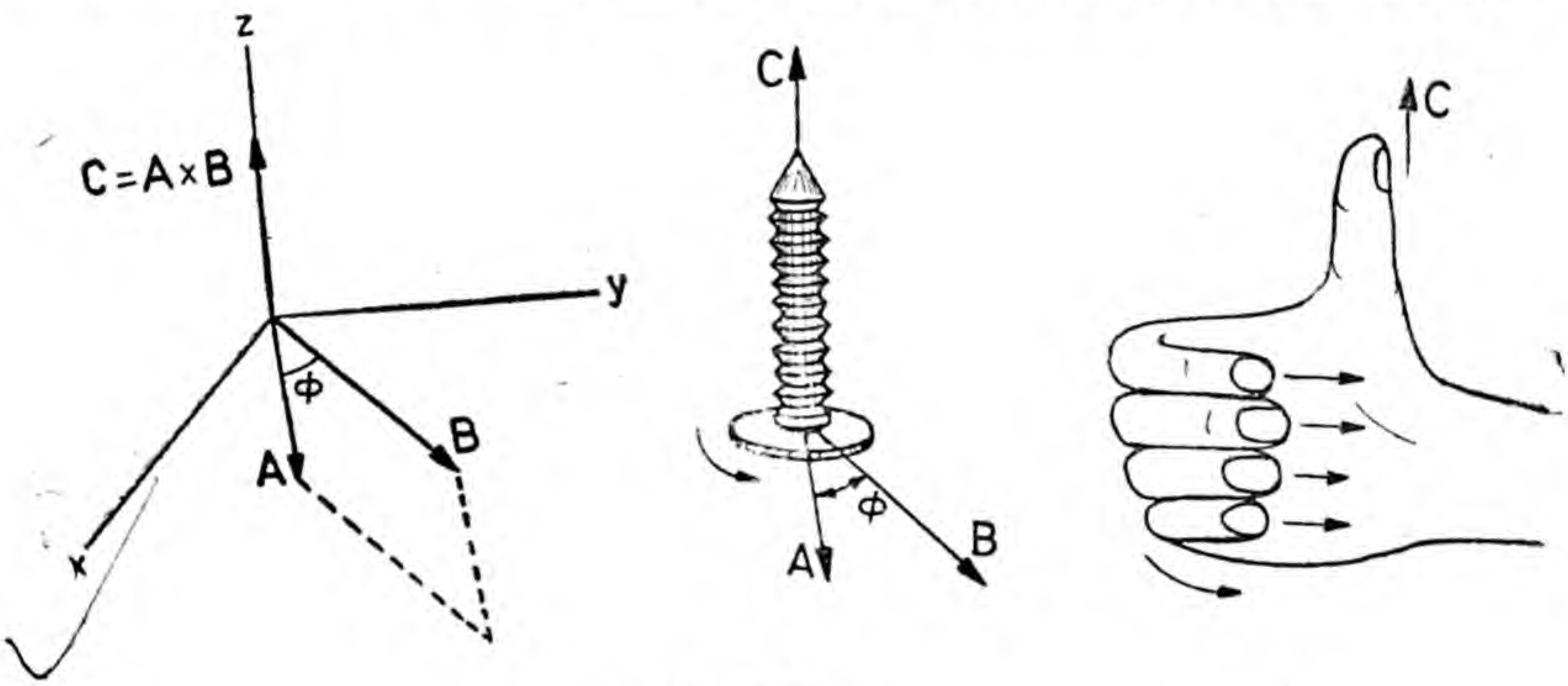


Fig. 1.5.

angle between them. *Its direction is ⊥ to the plane containing these vectors and is governed by the right handed screw if turned from the first vector towards the second vector.* If vectors **A** and **B** are in $x-y$ plane, the resultant vector $\mathbf{C}=\mathbf{A}\times\mathbf{B}$ is ⊥ to both **A** and **B** and is therefore in the z-direction.

$$\mathbf{C}=\mathbf{A}\times\mathbf{B}=AB\sin\phi\,\hat{n}=-\mathbf{B}\times\mathbf{A}. \qquad ...(14)$$

where $\hat{n}$ indicates the direction of normal.

Since $AB\sin\theta$ represents the area of the parallelogram having **A** and **B** as adjacent sides. Hence vector product of two vectors may be defined as a vector of magnitude equal to the area of the parallelogram with these vectors as its adjacent sides and in the direction ⊥ to this area (Fig 1.5).

Another way to obtain the direction of a vector product is the *right hand rule.* If we wrap fingers of the right hand around the axis, which is ⊥ to the plane of vectors **A** and **B**, so that the curled fingers follow the rotation of **A** into **B**, the extended right thumb will give the direction of the vector product $\mathbf{A}\times\mathbf{B}$.

Since $\mathbf{A}\times\mathbf{B}\neq\mathbf{B}\times\mathbf{A}$, hence the vector product is not *commutative.* Relation (14) shows that

$$\mathbf{A}\times\mathbf{B}=0 \qquad \text{if } \mathbf{A} \parallel \mathbf{B} \text{ or } \mathbf{A}=\mathbf{B}$$

$$\mathbf{A}\times\mathbf{B}=AB\hat{n} \qquad \text{if } \mathbf{A} \perp \mathbf{B}.$$

For unit vectors, we have

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

and $$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0. \qquad ...(15)$$

In cartesian co-ordinates, the vector product may be expressed as

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}=\mathbf{i}(A_yB_z-A_zB_y)+\mathbf{j}(A_zB_x-A_xB_z)+\mathbf{k}(A_xB_y-A_yB_x).$$

1.3. MULTIPLE PRODUCT

(*i*) $\mathbf{A}.(\mathbf{B}+\mathbf{C})=\mathbf{A}.\mathbf{B}+\mathbf{A}.\mathbf{C}$ ...(16)

(*ii*) $\mathbf{A}\times(\mathbf{B}+\mathbf{C})=\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}.$ ...(17)

(*iii*) $(\mathbf{A}.\mathbf{B})\mathbf{C}=(AB\cos\theta)\mathbf{C}=ABC\cos\theta$ along vector $\mathbf{C}$. ...(18)

(*iv*) $\mathbf{A}.(\mathbf{B}\times\mathbf{C})=\mathbf{B}.(\mathbf{C}\times\mathbf{A})=\mathbf{C}.(\mathbf{A}\times\mathbf{B})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$ ...(19)

It is a scalar quantity, equal in magnitude to the volume of the parallelopiped whose sides are formed from the three vectors **A**, **B**, and **C**. On account of this reason, such a vector product is called *scalar triple product*. It has following properties.

(*a*) If **A**, **B**, and **C** are coplaner or parallel to the same plane.

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C})=0.$$

(*b*) If **A**,**B**, and **C** are orthogonal to each other,

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C})=ABC.$$

(*c*) If any two of the three are identical or parallel.

$$\mathbf{A}.(\mathbf{A}\times\mathbf{C})=\mathbf{A}.(\mathbf{B}\times\mathbf{A})=\mathbf{A}.(\mathbf{B}\times\mathbf{B})=0.$$

(*v*) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}.\mathbf{C})\mathbf{B}-(\mathbf{A}.\mathbf{B})\,\mathbf{C}.$ ...(20)

It is a vector quantity and is thus called *vector triple product* One can prove very easily that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=0, \qquad ...(21)$$

1.4. DIFFERENTIATION AND INTEGRATION OF VECTORS

Vector quantities are often expressed as functions of scalar variables. For example, the electric field **E** can be expressed as a function of the position co-ordinates x, y and z. The vector may be differentiated and integrated with respect to these variables. Thus if $\mathbf{F}(u)$ is a vector function of the scalar variable u, the differential of **F** with respect to a scalar variable u is defined as

$$\frac{d\mathbf{F}}{du}=\lim_{\Delta u\to 0}\frac{\Delta\mathbf{F}}{\Delta u}=\lim_{\Delta u\to 0}\frac{\mathbf{F}(u+\Delta u)-\mathbf{F}(u)}{\Delta u}. \quad ...(22)$$

The derivative is a vector in the direction of $\Delta\mathbf{F}$ as Δu approaches zero. The derivative of the sum of two vectors is the sum of the derivatives. The derivative of a product is the same as that for the derivative of a product of scalars, with due regard for the change in sign if the order of the vectors is changed in the case of the vector product, such as :

If $\mathbf{C}=\mathbf{A}+\mathbf{B}$, then

$$\frac{d\mathbf{C}}{dt}=\frac{d\mathbf{A}}{dt}+\frac{d\mathbf{B}}{dt} \text{ or } \dot{\mathbf{C}}=\dot{\mathbf{A}}+\dot{\mathbf{B}}. \quad ...(23)$$

If $\mathbf{C}=b\mathbf{A}$, then

$$\frac{d\mathbf{C}}{dt}=\frac{db}{dt}\mathbf{A}+b\frac{d\mathbf{A}}{dt} \text{ or } \dot{\mathbf{C}}=\dot{b}\mathbf{A}+b\dot{\mathbf{A}}. \quad ...(24)$$

If $\mathbf{C}=\mathbf{A}\,.\,\mathbf{B}$, then

$$\frac{d\mathbf{C}}{dt}=\frac{d\mathbf{A}}{dt}.\,\mathbf{B}+\mathbf{A}.\frac{d\mathbf{B}}{dt} \text{ or } \dot{\mathbf{C}}=\dot{\mathbf{A}}\,.\,\mathbf{B}+\mathbf{A}\,.\,\dot{\mathbf{B}}. \quad ...(25)$$

If $\mathbf{C}=\mathbf{A}\times\mathbf{B}$, then

$$\frac{d\mathbf{C}}{dt}=\frac{d\mathbf{A}}{dt}\times\mathbf{B}+\mathbf{A}\times\frac{d\mathbf{B}}{dt} \text{ or } \dot{\mathbf{C}}=\dot{\mathbf{A}}\times\mathbf{B}+\mathbf{A}\times\dot{\mathbf{B}}. \quad ...(26)$$

We can differentiate the scalar triple product or vector triple product with the ordinary processes of differentiation, keeping the order of vectors as unchanged.

The process of integration is the limit approached in a summation of simple products. When a force $\mathbf{F}$ acts for a small distance $d\mathbf{l}$, the work done $dW=\mathbf{F}.d\mathbf{l}$. Total work done over a large distance is given by

$$W=\int\mathbf{F}\,.\,d\mathbf{l}=\int F\cos\theta\,dl. \quad ...(27)$$

This integral is known as line integral of $\mathbf{F}$ along the curve. Similarly we have surface and volume integrals.

1.5. SCALAR AND VECTOR FIELDS

A physical quantity which can be expressed as a continuous function of the position of the point in a region of space is known as a *point or position function*. The region in which it specifies the physical quantity is known as a *field*. A scalar function ϕ which has values throughout a region constitutes a field known as *scalar field*. Examples are the density fields or temperature fields. Vector function **F** which has values throughout a region constitutes a field known as *vector field*. Examples are the *velocity field* or the *force field*.

1.6. THE GRADIENT OF A SCALAR FIELD

Let ϕ be a scalar point function, which is continuous in the given region of space. In this region, consider two level surfaces (the level surface of a scalar point function is a surface for all points of which the function has the same value, *e.g.*, isothermal or equipotential surfaces) S_1 and S_2 very close together, Fig 1.6. If these surfaces are characterised by scalar functions ϕ and $\phi+\delta\phi$ respectively. Reference to the origin O, we then have

$$\mathbf{OP}=\mathbf{r}, \quad \mathbf{OP}'=\mathbf{r}+\delta\mathbf{r},$$

hence $\mathbf{PP}'=\delta\mathbf{r}$.

The rate of increase of ϕ at point P in the direction $\mathbf{PP}'$ is $\delta\phi/\delta r$ with the limit $\delta r \to 0$. The least distance

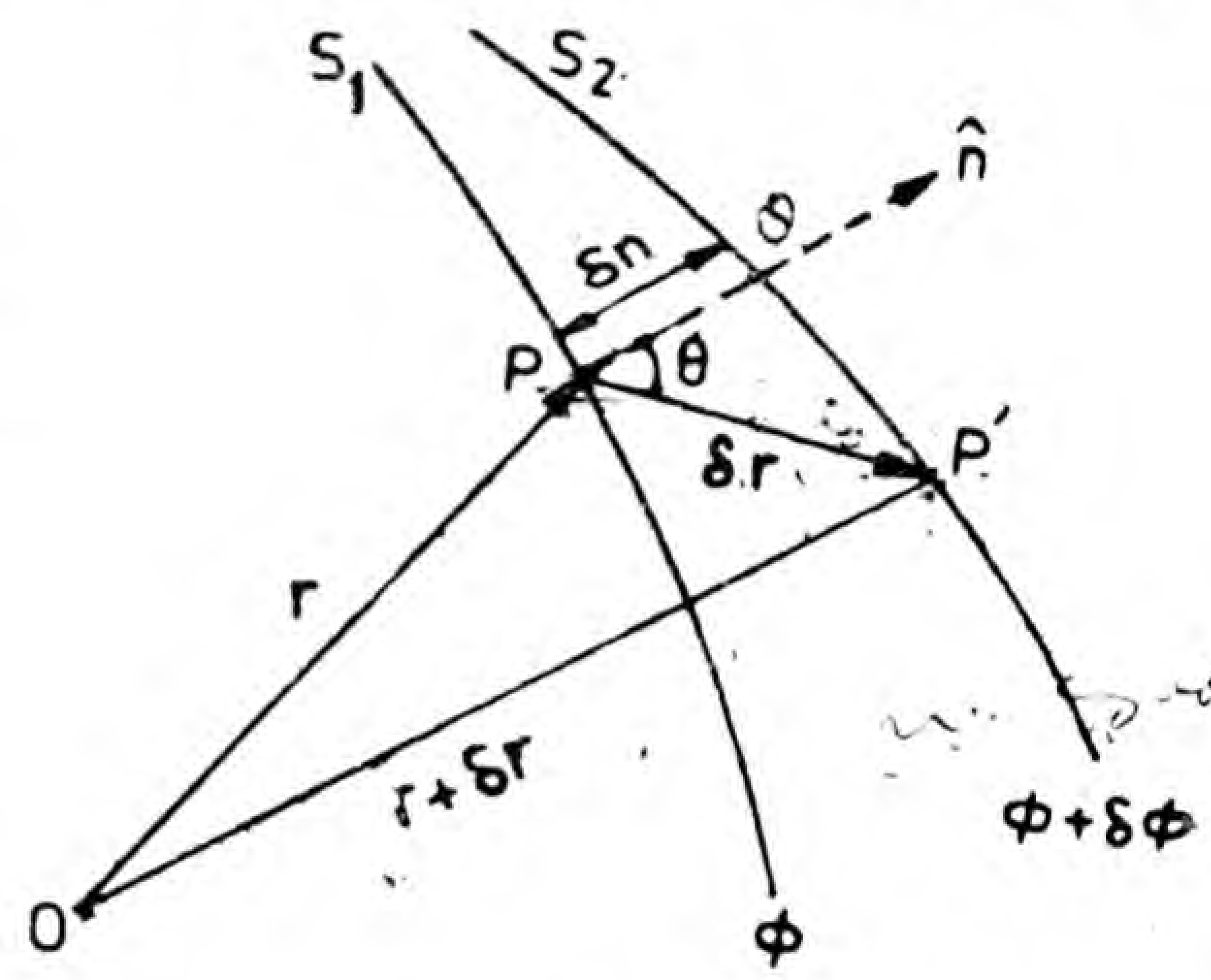


Fig. 1.6

between the surfaces is PQ in the direction of unit normal vector $\hat{n}$. Here $\hat{n}$ represents a unit normal at P to the level surface of ϕ through P in the direction of ϕ increasing. Thus the maximum rate of increase of ϕ is $\partial\phi/\partial n$ and is along $\hat{n}$. If the normal at P makes an angle θ with $\mathbf{PP}'$, then we have

$$\delta n=\delta r\cos\theta \text{ or } \delta n=\hat{n}\,.\,\delta\mathbf{r}.$$

$$\therefore \quad \frac{\delta\phi}{\delta r}=\frac{\delta\phi}{\delta n}\cdot\frac{\delta n}{\delta r}=\frac{\delta\phi}{\delta n}\cos\theta,$$

As $\cos\theta \leqslant 1$, hence $\delta n \leqslant \delta r$ and $\delta\phi/\delta r < \delta\phi/\delta n$,

or
$$\lim_{\delta r\to 0}\frac{\delta\phi}{\delta r} \leqslant \frac{\partial\phi}{\partial n}. \qquad \ldots(28)$$

Thus the vector $(\partial\phi/\partial n)\,\hat{n}$ gives the greatest rate of increase of ϕ at the point P and is known as *gradient* or *grad of a scalar point function* ϕ. Hence

$$\text{grad}\ \phi=(\partial\phi/\partial n)\,\hat{n}. \qquad \ldots(29)$$

It shows that the gradient of a scalar field is a *vector field*. This vector has magnitude equal to the most rapid rate of change of the scalar field and is in the direction in which this rate is maximum.

As $$\frac{\partial \phi}{\partial n}=\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial n}+\frac{\partial \phi}{\partial y}\frac{\partial y}{\partial n}+\frac{\partial \phi}{\partial z}\frac{\partial z}{\partial n}.$$

$$\therefore \quad \text{grad}\,\phi=\frac{\partial \phi}{\partial n}\hat{n}=\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial n}\hat{n}+\frac{\partial \phi}{\partial y}\frac{\partial y}{\partial n}\hat{n}+\frac{\partial \phi}{\partial z}\frac{\partial z}{\partial n}\hat{n}$$

$$=\frac{\partial \phi}{\partial x}\mathbf{i}+\frac{\partial \phi}{\partial y}\mathbf{j}+\frac{\partial \phi}{\partial z}\mathbf{k}, \qquad ...(30)$$

where $(\partial x/\partial n)\,\hat{n}$, $(\partial y/\partial n)\,\hat{n}$ and $(\partial z/\partial n)\,\hat{n}$ are the unit vectors **i**, **j** and **k** along the axes x, y and z respectively.

$$\therefore \text{ grad}\,\phi=\left[\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{5\partial}{\partial z}\right]\phi=\nabla\phi, \qquad ...(31)$$

where the differential operator ∇ is pronounced as **del or nabla.** It is defined as

$$\nabla\equiv\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}. \qquad ...(32)$$

As a differential operator the ∇ vector by itself can be characterized neither by magnitude nor by direction. It only assumes a definite value when it is applied to some field quantity. If it is applied to a scalar ϕ, then we have $\nabla\phi\equiv\text{grad}\,\phi$. If the scalar function ϕ represents temperature, then $\nabla\phi=\text{grad}\,\phi$ is a temperature gradient. It is a vector quantity. Its direction being that in which the temperature changes most rapidly.

Important Deductions

(1) Gradient of sum of two scalar functions is the vector sum of their gradients, *i.e.*,

$$\nabla(u+v)=\nabla u+\nabla v.$$

(2) $\nabla(uv)=\nabla uv+v\nabla u.$

(3) $\nabla f(u)=f'(u)\nabla u.$ ...(33)

1.7. THE DIVERGENCE OF A VECTOR FIELD

For a physical concept of the divergence of a vector point function, **v**, let us consider an infinitesimal element of volume with sides Δx, Δy and Δz parallel to the axes of x, y and z, respectively

(Fig 1.7). Let the vector point function **v** at the middle of this element has components of magnitude v_x, v_y and v_z along these axes.

Consider the two faces of the volume element each with area $\Delta x\, \Delta z$ perpendicular to the y-axis. On the left hand face of the volume, the value of v_y at the middle of the face becomes

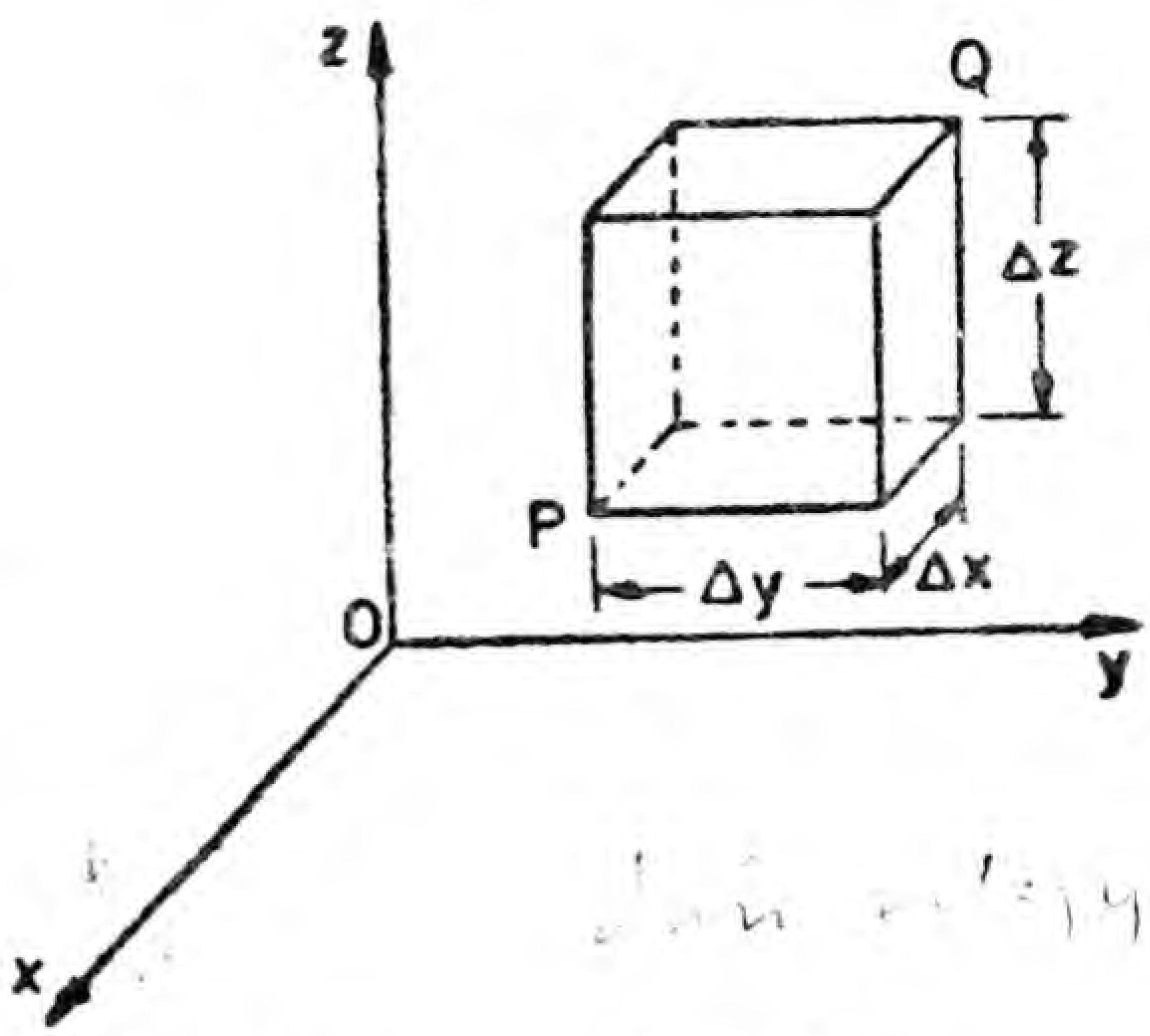


Fig. 1.7.

$$v_y - \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y.$$

Since the face area is infinitesimally small, hence the above value may be taken as the value all over the face. Similarly, on the right hand face, the value of v_y is

$$v_y + \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y.$$

We define the flux of a vector field through any face as the scalar product of the vector area of the face and the vector **v**, *i.e.*, the product of the area of the face and the normal component of the vector upon it. Thus the flux entering the element in the y-direction

$$= \left(v_y - \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y \right)\Delta x \Delta z.$$

and flux leaving the element in the y-direction

$$= \left(v_y + \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y \right)\Delta x \Delta z.$$

∴ Excess of flux leaving the element over that entering it in the y-direction is

$$\left(v_y + \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y \right)\Delta x \Delta z - \left(v_y - \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y \right)\Delta x \Delta z$$

$$= \frac{\partial v_y}{\partial y}\Delta x \Delta y \Delta z,$$

Similarly the net outward flux in the x-direction

$$= (\partial v_x/\partial x)\Delta x \Delta y \Delta z.$$

and in the z-direction, it is

$$= (\partial v_z/\partial z)\Delta x \Delta y \Delta z.$$

∴ Total net outward flux diverging from or leaving the element

$$=\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\Delta x\Delta y\Delta z$$

The amount of this flux per unit volume is defined as the divergence of the vector point function **v** and is thus written as

$$\text{div } \mathbf{v}=\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}. \qquad \ldots(34)$$

This expression does not contain the dimensions of the box. Since the divergence is the amount of net flux, it is essentially a scalar.

Since

$$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right).(\mathbf{i}A_x+\mathbf{j}A_y+\mathbf{k}A_z)$$

$$=\nabla.\mathbf{A}.$$

Hence $\text{div } \mathbf{v}=\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=\nabla.\mathbf{v}.$ ...(35)

Since the operator ∇ is invariant, *i.e.* independent of any system of axes, hence the divergence of any vector is also *invariant*. If the divergence exists at a point in a fluid of density ρ_m, the quantity $[\nabla.(\rho_m\mathbf{v})]$ of fluid is known as the strength of the **v** flow or simply the flux of **v**. This represents the excess of the outward over the inward flow or the divergence of the fluid. The quantity $-\nabla.(\rho_m\mathbf{v})$ represents the excess of the inward over the outward flow, or the convergence of the fluid. Due to this reason $\nabla.(\rho_m\mathbf{v})$ or $\nabla.\mathbf{F}$ is named divergence of $\rho_m\mathbf{v}$ or **F**, denoted by *div*.

In general we can write it for a vector function **F** as

$$\text{div}\mathbf{F}=\nabla.\ \mathbf{F}=\partial F_x/\partial x+\partial F_y/\partial y+\partial F_z/\partial z. \qquad \ldots(36)$$

It is a scalar quantity and can be represented as

$$\nabla.\mathbf{F}=\Sigma\mathbf{i}.\ (\partial\mathbf{F}_x/\partial x). \qquad \ldots(37)$$

Equation (36) can be written as

$$\text{div } \mathbf{F}=\lim_{V\to 0}\frac{1}{V}\int\mathbf{F}.\ d\mathbf{S}. \qquad \ldots(38)$$

Thus the divergence of a vector is the limit of its surface integral per unit volume as the volume V enclosed by the surface goes to zero. It is a scalar quantity. It represents the net amount of flux coming out of a volume element.

The divergence of a field is positive at any point if either the fluid is expanding and its density is decreasing at that point or there is a *source of fluid.* The divergence is negative if the fluid is contracting and its density is increasing or the point is a *sink*. If the net flow of fluid is zero, divergence is zero.

If there is no source or sink and the density of the fluid is not changing, the fluid is called *incompressible.* The vector field satisfying this condition is said to be *solenoidal field.* Hence for incompressible fluid or for a solenoidal field, divergence of a vector field **F** is always zero, *i.e.*, div $\mathbf{F}=\nabla \cdot \mathbf{F}=0$.

The flux may be flow of liquid, flow of heat or electric flux. Thus in the electric flux, the existence of a finite positive value of the divergence at a point shows that there must be a positive charge at the concerned point.

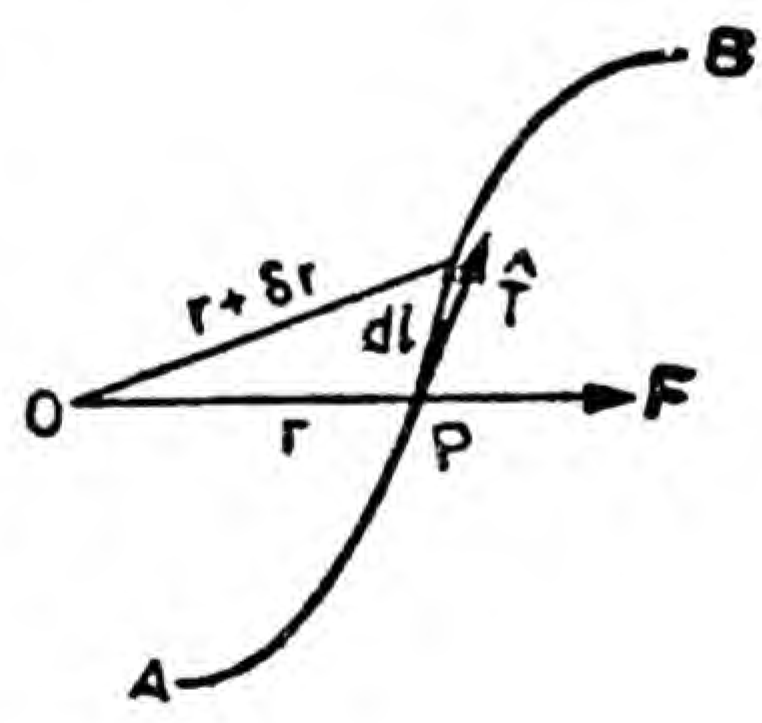
1.8. LINE INTEGRAL

Let $\mathbf{F}(x,y,z)$ be a vector function defined throughout some region of space with two positions A and B anywhere in this region (Fig 1.8). The line integral of **F** along the curve of some path that runs between A and B is defined as the integral of tangential component of **F** along the curve. Thus

$$\text{line integral}=\int_A^B \mathbf{F}\cdot d\mathbf{l}. \qquad \text{...(39)}$$

This means : Divide the path into short segments, each segment is represented by a vector $d\mathbf{l}$ and take the scalar product of the path segment vector with the vector **F**. Now add these products for the whole path. If the segments are made shorter, this sum can be represented by the integral from one end to other.

If $\hat{T}$ is a unit vector tangent to the curve at any point P with a position vector **r**. Its value is given as



$$\hat{T}=d\mathbf{r}/dl.$$

Fig. 1.8

$$\therefore \quad \text{Line integral}=\int \mathbf{F}\cdot d\mathbf{l}=\int \mathbf{F}\cdot \hat{T}\, dl=\int \mathbf{F}\cdot \frac{d\mathbf{r}}{dl}\cdot dl$$

$$=\int \mathbf{F}\cdot d\mathbf{r} \qquad (\text{...40})$$

Thus we see that $d\mathbf{l}$ is equivalent to $d\mathbf{r}$. For the three dimensional case, we have

$$\mathbf{F}=F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k} \text{ and } d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}.$$

$$\therefore \quad \int \mathbf{F}\cdot d\mathbf{r}=\int (F_x dx+F_y dy+F_z dz). \qquad \text{...(41)}$$

The line integral may or may not depend upon the path of integration. When it is independent of integration, the field defined by **F** is known as **conservative field.** In this case

$$\int_{C_1} \mathbf{F}.\, d\mathbf{r} = \int_{C_2} \mathbf{F}.d\mathbf{r}, \qquad \text{...(42)}$$

where C_1 and C_2 are the two curves with the same end points A and B. If we travel from A to B through curve C_1 and come back to A through C_2, then for conservative field **F**, we get

$$\oint \mathbf{F}.d\mathbf{r} = \int_{C_1} \mathbf{F}.d\mathbf{r} + \int_{C_2} \mathbf{F}.d\mathbf{r} = 0. \qquad \text{...(43)}$$

Let $\mathbf{F} = \nabla\phi$, therefore

$$\int_B^A \mathbf{F}.d\mathbf{r} = \int_A^B \nabla\phi.d\mathbf{r} = \int_A^B d\phi = \phi_B - \phi_A, \qquad \text{...(44)}$$

which is independent of path but depends only on the end points, A and B.

For any closed path the above relation reduces to

$$\oint \mathbf{F}.d\mathbf{r} = \int_A^B \mathbf{F}.d\mathbf{r} + \int_B^A \mathbf{F}.d\mathbf{r}$$

$$= \phi_B - \phi_A + \phi_A - \phi_B = 0. \qquad \text{...(45)}$$

Thus a conservative field is that field which may be defined as a gradient of a scalar field and for which the closed line integral is always zero. Such fields are called *irrotational fields*. As these are obtained from the scalar by taking gradients, hence are called *scalar potential fields*. Since space is divided up into layers or laminae by the level or equipotential surfaces of the function ϕ, hence the fields are also called the *lamellar vector fields*.

1.9. SURFACE INTEGRAL

If **F** represents a velocity vector **v** of a fluid, the dot product $\mathbf{v}.\delta\mathbf{S}$ represents the rate of flow of fluid through an area $\delta\mathbf{S}$ (whose direction is perpendicular to the surface) and is also known as *normal velocity flux*. Consider some vector field **F** in space and in this space some arbitrary closed surface. Now divide the whole surface into little patches which are so small that over any one patch the surface is practically flat and the vector field is same at all points on it (Fig 1.9). The dot product $\mathbf{F}.\,\delta\mathbf{S}$ is the flux through the patch of surface area $\delta\mathbf{S}$. If we add all these products for all patches we get the flux Φ through the entire surface.

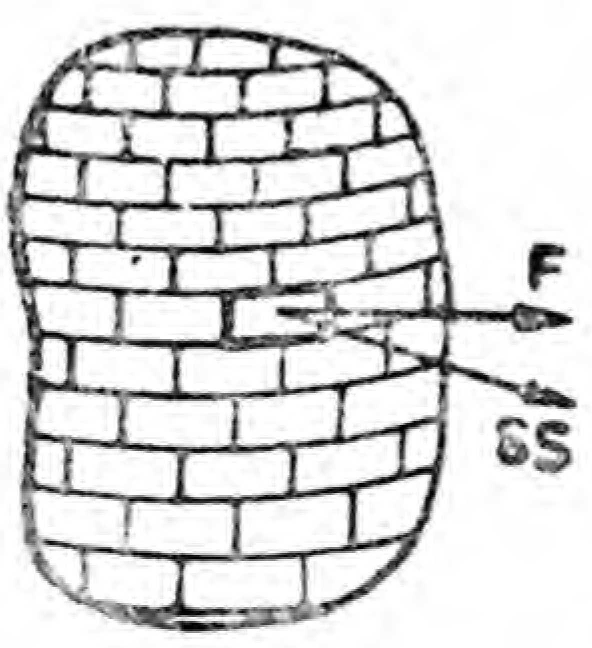


Fig. 1.9

$$\mathbf{\Phi} = \Sigma \mathbf{F}.\delta\mathbf{S}. \qquad \text{...(46)}$$

If the patches are of very small area, the flux Φ can be written as the surface integral, *i.e.*,

$$\Phi=\int_{\text{Entire surface}} \mathbf{F}.d\mathbf{S}. \qquad ...(47)$$

Also the surface is a two-dimensional quantity, we generally use double integral, such as

$$\Phi=\iint_S \mathbf{F}.d\mathbf{S}. \qquad ...(48)$$

Since $d\mathbf{S}=\mathbf{i}dS_x+\mathbf{j}dS_y+\mathbf{k}dS_z$, hence we have

$$\Phi=\iint (F_x dS_x+F_y dS_y+F_z dS_z). \qquad ...(49)$$

1.10. GAUSS'S DIVERGENCE THEOREM

This theorem states that *the surface integral of the normal component of a vector point function* **F** *taken over a closed surface S enclosing a volume V is equal to the volume integral of the divergence of the vector* **F** *taken through the volume V*, *i.e.*,

$$\iint_S \mathbf{F}.d\mathbf{S}=\iiint_V \text{div}\,\mathbf{F}dV. \qquad ...(50)$$

In rectangular co-ordinates

$$\mathbf{F}=F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k},\ dV=dx\,dy\,dz,$$
$$dS_x=dy\,dz,\ dS_y=dx\,dz \text{ and } dS_z=dx\,dy.$$

The equation (50) may thus be written as

$$\iint_S (F_x dS_x+F_y dS_y+F_z dS_z)=\iiint_V \left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right) dx\,dy\,dz \qquad ...(51)$$

To prove this theorem, let us first integrate the second term of right hand side with respect to y. For this we consider a rectangu-

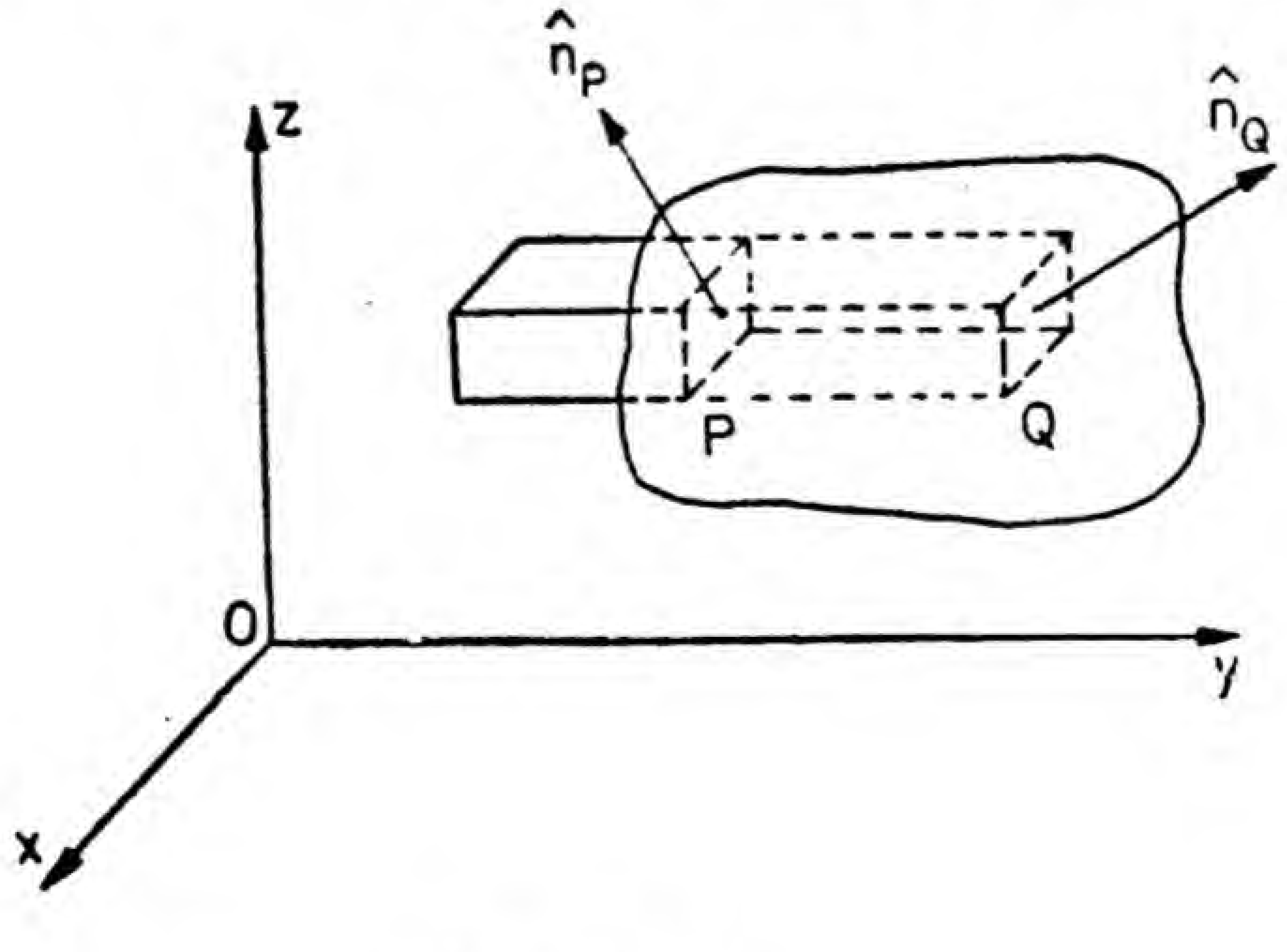


Fig. 1.10.

lar prism PQ with cross sectional area $dx.dz$ and edges parallel to y-axis. We thus get

$$\iiint_V \frac{\partial F_y}{\partial y} dx.dy.dz = \iint \Big[F_y(x, y_2, z) - F_y(x, y_1, z) \Big] dxdz \qquad ...(52)$$

Here x, y_1, z are the co-ordinates of end P and x, y_2, z of the end. Q. Let $\hat{n}_p$ and $\hat{n}_q$ be the unit vectors along the normals to the elements of areas dS_P and dS_Q, the areas where the elementary prism PQ meets the close surface. The area dS has a projection on the x-z plane of value $dS_y = dx.\ dz$. As $\hat{n}_p$ makes an obtuse angle with the y axis, hence the projection of dS_P on x-z plane

$$= (-\mathbf{i}.\hat{n}_p)\ dS_P = dx.\ dz = -(dS_y)_P.$$

Similarly for the area dS_Q the projection on x-z plane

$$= (\mathbf{i}.\hat{n}_Q) dS_Q = dx.dz = (dS_y)_Q.$$

Hence the equation (52) becomes

$$\iiint_V \frac{\partial F_y}{\partial y} dx.dy.dz = \iint \Big[(F_y dS_y)_Q + (F_y dS_y)_P \Big].$$

Thus dividing the whole volume into the large number of rectangular prisms with edges parallel to y-axis and of small cross sectional areas. Hence on adding all contributions we get equation (51) as

$$\iiint_V \frac{\partial F_y}{\partial y} dy.dx.dz = \iint F_y dS_y. \qquad ...(53)$$

Similarly we can show that

$$\iiint_V \frac{\partial F_x}{\partial x} dx.\ dy.\ dz = \iint F_x dS_y \qquad ...(54)$$

and
$$\iiint_V \frac{\partial F_z}{\partial z} dz.dx.dy = \iint F_z dS_z. \qquad ...(5)$$

On adding these equations, we get

$$\iiint_V \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx.\ dy.dz.$$

$$= \iint_S (F_x dS_x + F_y dS_y + F_z dS_z).$$

This is *Gauss's divergence theorem.* It is very useful when we transform the surface integral of vector function over a closed surface into a volume integral. This theorem also shows that if the surface integral of a vector **F** is equal to the volume integral of a scalar function P over the volume enclosed by the surface, then we get

$$P = \text{div } \mathbf{F}. \qquad ...(56)$$

1.11. ROTATIONAL VECTOR FIELDS, CURL OF A VECTOR

We have read that for irrotational fields the line integral around a closed path is zero. There are also other vector fields for which the line integral around a closed path is not zero, such fields are known as *rotational vector fields*.

The curl is an important property for these vector fields. Let us first understand the *physical significance* of a curl of a vector field.

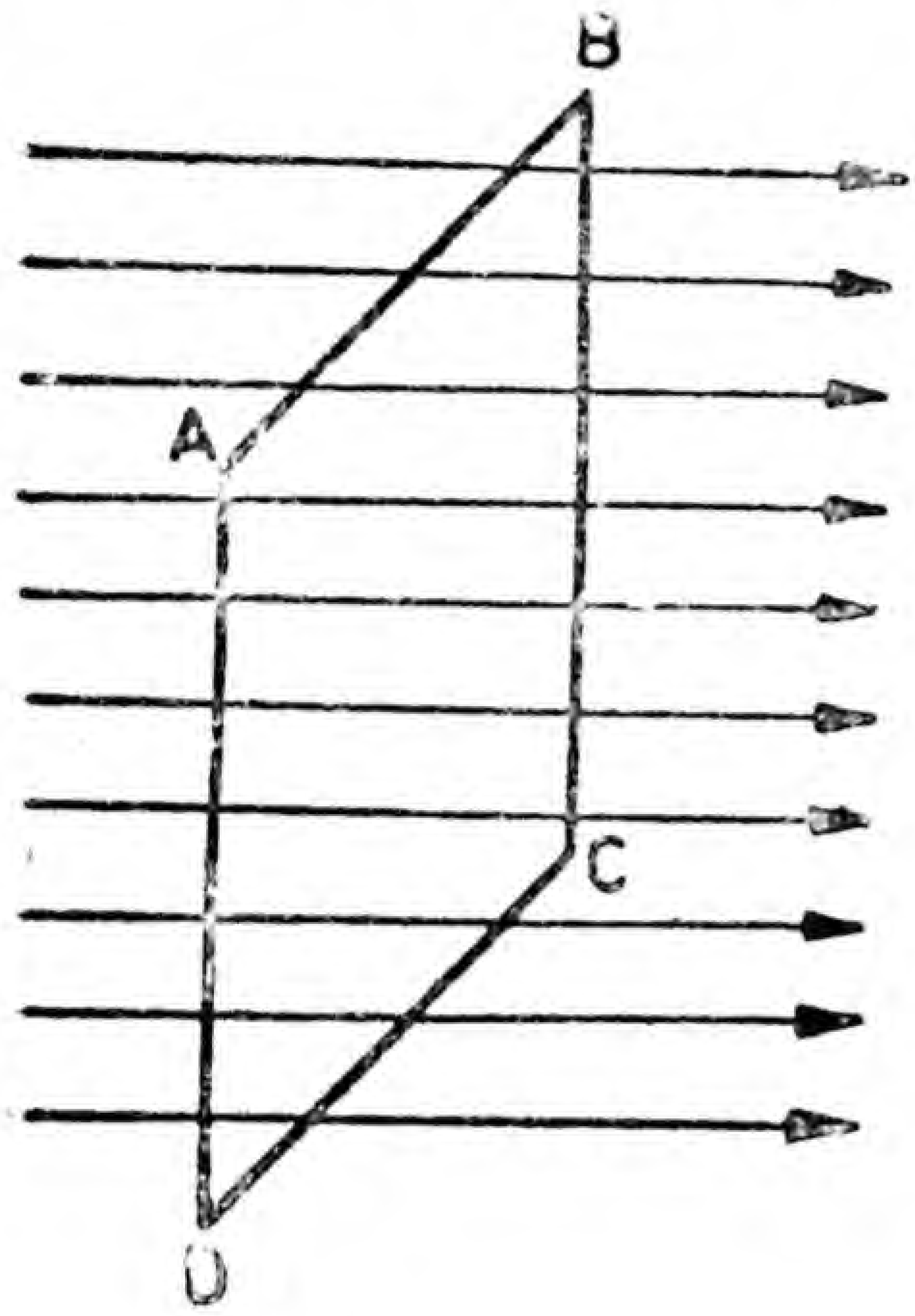


Fig. 1.11

Consider a small rectangular plane $ABCD$ placed in a very small region of rotational vector field. When the plane is in such a way that the vector field is normal to it, *i.e.*, normal to each side of this plane, the line integral along its all sides is

$$\oint \mathbf{F}.d\mathbf{l} = \int_A^B \mathbf{F}.d\mathbf{l} + \int_B^C \mathbf{F}.d\mathbf{l} + \int_C^D \mathbf{F}.d\mathbf{l} + \int_D^A \mathbf{F}.d\mathbf{l}$$
$$= 0+0+0+0=0. \quad \ldots(57)$$

If this plane is rotated to 90° such that the vector field is parallel to it, *i.e.*, to the sides AB and CD then the line integral

$$\oint \mathbf{F}.d\mathbf{l} = \int_A^B \mathbf{F}.d\mathbf{l} + \int_B^C \mathbf{F}.d\mathbf{l} + \int_C^D \mathbf{F}.d\mathbf{l} + \int_D^A \mathbf{F}.d\mathbf{l}$$
$$= \int_A^B \mathbf{F}.d\mathbf{l} - \int_C^D \mathbf{F}.d\mathbf{l}. \quad \ldots(58)$$

As **F** is continuously varying with the position, hence **F** for AB is different than that for CD and the net closed line integral is of a finite value. Thus the closed line integral along $ABCD$ depends upon its orientation with respect to the vector field. There is a certain orientation of the area for which the line integral is maximum. This maximum line integral at any point in such a vector field around a closed curve, expressed for unit area is called the **curl of a vector field** at that point. It is a vector quantity directed along the normal to the test area, towards which a right handed screw moves when turned in the sense in which the line integral along the boundary of the test area is maximum. Hence curl **F** is defined as

$$\text{curl } \mathbf{F} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint \mathbf{F}.d\mathbf{l}. \quad \ldots(59)$$

To understand clearly the curl of a vector and its direction, let us consider a small paddle wheel in the path of the flow of water in a canal. On account of the viscous property of the water, the velocity of the water layer decreases downwards, it is nearly zero at the bottom and is maximum at the top. The wheel will thus turn

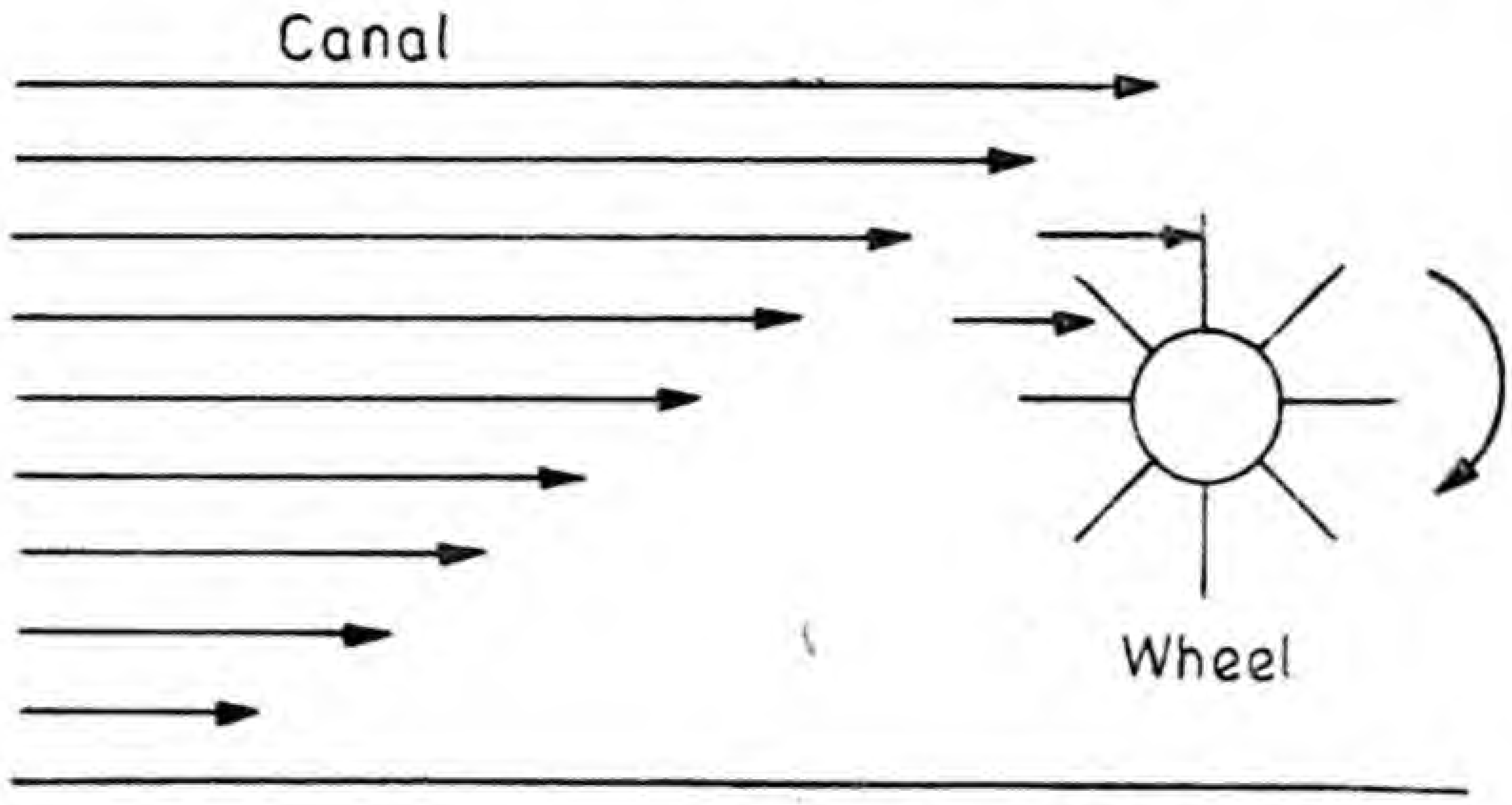


Fig. 1.12.

in the clockwise direction, as shown in Fig. 1.12, showing that there is a small circulation of water round about the wheel. The axis about which the wheel rotates gives the direction of the **curl**.

Let us find out the curl of a vector field **F** in terms of the cartesian components. Consider a rectangular area *ABCD* perpendicular to the axis of *y*, having sides $\triangle x$ and $\triangle z$. (Fig 1.13). Because of the small sides we can assume that the numerical value of the component of **F** at the middle point of any side is the average value along that side. Let F_x, F_y, and F_z be the components of vector **F** along *x*-, *y*-, and *z*- axes respectively at the origion O. Thus average values along sides *AB*, *BC*, *CD* and *DA* are

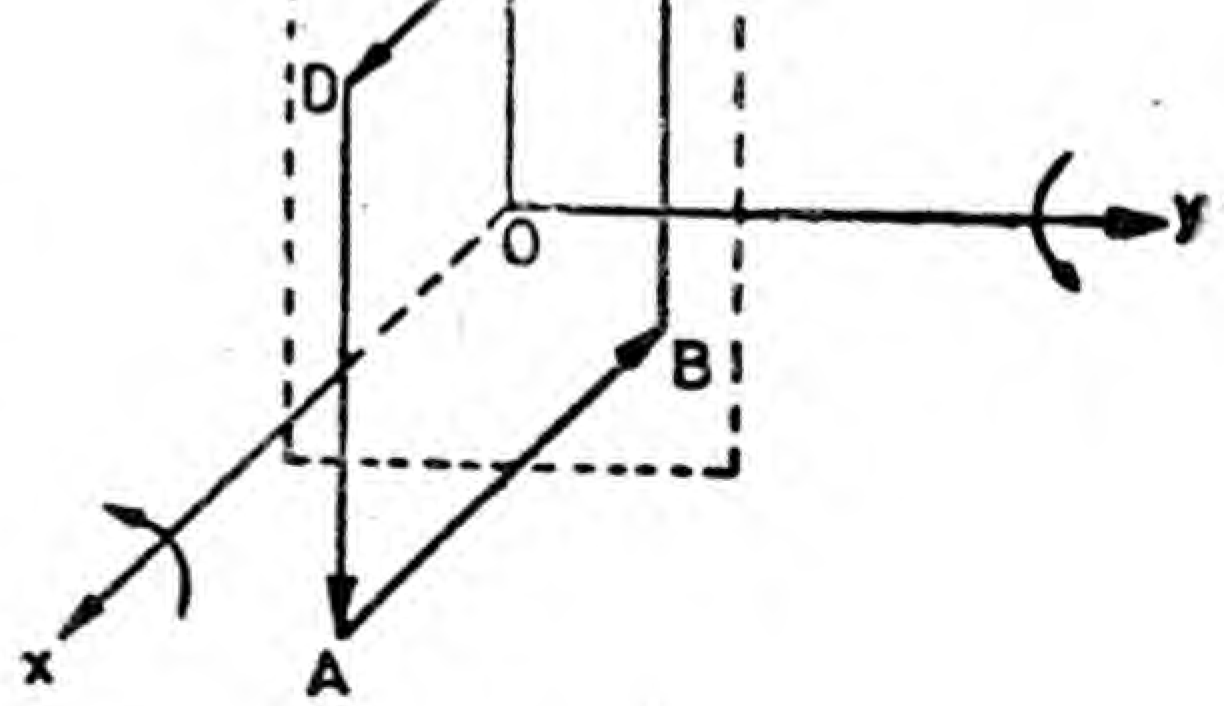


Fig. 1.13.

$$\left[F_x+\frac{\partial F_x}{\partial z}(-\tfrac{1}{2}\triangle z)\right], \left[F_z+\frac{\partial F_z}{\partial x}(-\tfrac{1}{2}\triangle x)\right],$$

$$\left[F_x+\frac{\partial F_x}{\partial z}(\tfrac{1}{2}\triangle z)\right] \text{ and } \left[F_z+\frac{\partial F_z}{\partial x}(\tfrac{1}{2}\triangle x)\right] \text{ respectively.}$$

As $$\int_A^B \mathbf{F}.d\mathbf{l}=\left[F_x+\frac{\partial F_x}{\partial z}(-\tfrac{1}{2}\triangle z)\right](-\triangle x).$$

$$\int_B^C \mathbf{F}.d\mathbf{l}=\left[F_z+\frac{\partial F_z}{\partial x}(-\tfrac{1}{2}\Delta x)\right](\Delta z),$$

$$\int_C^D \mathbf{F}.d\mathbf{l}=\left[F_x+\frac{\partial F_x}{\partial z}(\tfrac{1}{2}\Delta z)\right](\Delta x),$$

and
$$\int_D^A \mathbf{F}.d\mathbf{l}=\left[F_z+\frac{\partial F_z}{\partial x}(\tfrac{1}{2}\Delta x)\right](-\Delta z).$$

Hence the total line integral of **F** along the boundary *ABCD*

$$\oint \mathbf{F}.d\mathbf{l}=\int_A^B \mathbf{F}.d\mathbf{l}+\int_B^C \mathbf{F}.d\mathbf{l}+\int_C^D \mathbf{F}.d\mathbf{l}+\int_D^A \mathbf{F}.d\mathbf{l}$$

$$=(\partial F_x/\partial z-\partial F_z/\partial x)\Delta x\Delta z.$$

The value of this integral per unit area is $(\partial F_x/\partial z-\partial F_z/\partial x)$. It is in the direction normal to the area *ABCD*, *i.e.* in the y-direction. It is thus the y-component of the curl of a vector function **F**. Hence

$$(\text{curl } \mathbf{F})_y=(\partial F_x/\partial z-\partial F_z/\partial x)\ \mathbf{j},$$

where **j** is the unit vector along y-axis.

Similarly we have

$(\text{curl } \mathbf{F})_x=(\partial F_z/\partial y-\partial F_y/\partial z)\ \mathbf{i}$ and $(\text{curl } \mathbf{F})_z=(\partial F_y/\partial x-\partial F_x/\partial y)\mathbf{k}$.

$\therefore$ curl $\mathbf{F}=(\text{curl } \mathbf{F})_x+(\text{curl } \mathbf{F})_y+(\text{curl } \mathbf{F})_z$

$$=\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{k}. \qquad \ldots(60)$$

This may be conveniently written in the form of a determinant as

$$\text{curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \qquad \ldots(61)$$

Since the vector product $\nabla\times\mathbf{F}$ of the operator ∇ and the vector **F** may be written as

$$\nabla\times\mathbf{F}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k})$$

$\therefore$ curl $\mathbf{F}=\nabla\times\mathbf{F}=$ Del cross **F**.

Since the operation is independent of the system of the curl of a vector is *invariant*.

The name curl suggests that a vector field **F** at a point of space has circulation there. As the curl is associated in hydrodynamics with the rotation of a fluid hence is sometimes called *rotation* or simply *rot*. *For any conservative vector field the curl is zero at all points of space, as the line integral for a closed path is zero for such a field.*

1.12. STOKE'S THEOREM

This theorem states that *the line integral of tangential component of a vector field* **F** *around a closed curve is equal to the surface integral of the normal component of its curl over the surface S bounded by that curve C, i.e.*

$$\oint_C \mathbf{F}.d\mathbf{l}=\oint_S \text{curl } \mathbf{F}.d\mathbf{S}. \qquad ...(63)$$

Consider a surface area $d\mathbf{S}$ at any point P on the surface S having closed boundary C (Fig 1.14 a). If $\hat{n}$ is the únit normal vector on this area, then $d\mathbf{S}=\hat{n}dS$ and the Stoke's theorem may also be expressed as

$$\int_C \mathbf{F}.d\mathbf{l}=\iint_S \hat{n}.\,(\text{curl}\mathbf{F})\,dS.$$

Let the surface S be subdivided by sets of curves lying on the surface and joining the points on the curve C so as to form a net work as shown in (Fig 1.14 b). If we consider the line integrals of **F** around the two adjacent meshes we see that the line integrals along

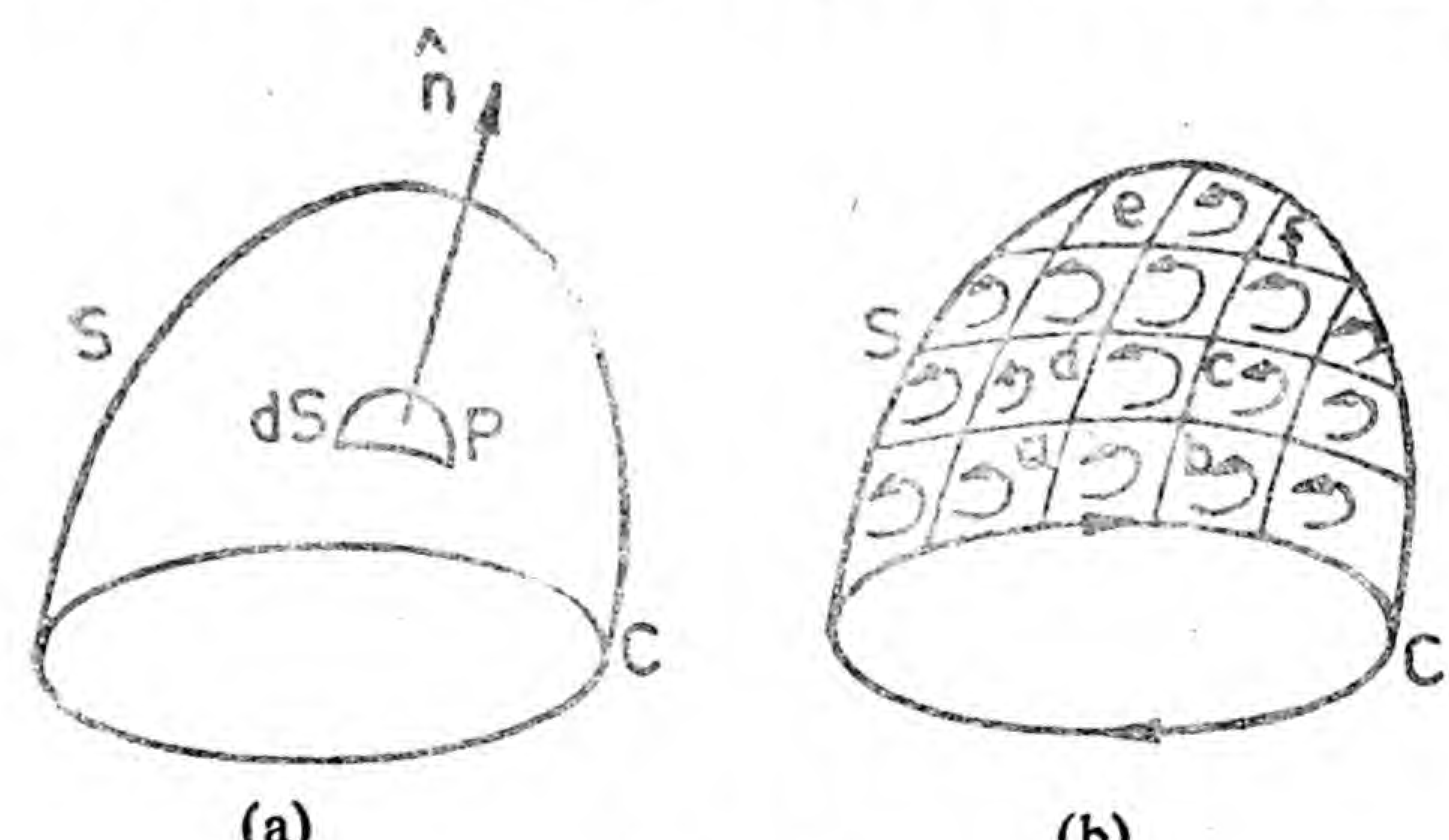


Fig. 1.14

the common side of the two meshes cancel each other as one has to travel in opposite directions. In the case of meshes *abcd* and *dcfe*, the common side *dc* does not contribute to the line integral and the line integral along *abcd*+line integral along *dcfe*=line integral along *abfe*. In general every mesh has a side common with its neighbouring meshes, hence the sum of the line integrals of **F** around all the meshes forming a network on the whole surface S is the line integral of **F** along the contour of the curve C, *i.e.*

$$\int_C \mathbf{F}.d\mathbf{l} = \sum_{\text{all meshes}} \oint \mathbf{F}.d\mathbf{l}. \qquad \ldots(64)$$

Let us consider a single mesh $ABCD$ which is a parallelogram with the sides AB and AD represented by infinitesimal vectors $\delta\mathbf{a}$ and $\delta\mathbf{b}$. Let the mean values of vector field **F** along the sides AB, BC, CD and DA be $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ respectively, then

$$\oint \mathbf{F}.d\mathbf{l} = \int_A^B \mathbf{F}.d\mathbf{l} + \int_B^C \mathbf{F}.d\mathbf{l} + \int_C^D \mathbf{F}.d\mathbf{l} + \int_D^A \mathbf{F}.d\mathbf{l}$$

$$= \mathbf{F}_1.\delta\mathbf{a} + \mathbf{F}_2.\delta\mathbf{b} - \mathbf{F}_3.\delta\mathbf{a} - \mathbf{F}_4.\delta\mathbf{b}$$

$$= \delta\mathbf{b}.(\mathbf{F}_2 - \mathbf{F}_4) - \delta\mathbf{a}.(\mathbf{F}_3 - \mathbf{F}_1). \qquad \ldots(65)$$

We know that the change in **F** corresponding to a displacement $\delta\mathbf{r}$ is given by

$$\delta\mathbf{F} = \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z}\right)\mathbf{F} = (\delta\mathbf{r} . \nabla)\,\mathbf{F}. \qquad \ldots(66)$$

Hence we can write

$$\mathbf{F}_2 - \mathbf{F}_4 = (\delta\mathbf{a}.\nabla)\,\mathbf{F}.$$

and

$$\mathbf{F}_3 - \mathbf{F}_1 = (\delta\mathbf{b}.\nabla)\,\mathbf{F}. \qquad \ldots(67)$$

Substituting these values in equation (65), we get

$$\oint \mathbf{F}.d\mathbf{l} = \delta\mathbf{b}.(\delta\mathbf{a}.\nabla)\mathbf{F} - \delta\mathbf{a}.(\delta\mathbf{b}.\nabla)\,\mathbf{F}$$

$$= [\delta\mathbf{b}(\delta\mathbf{a}.\nabla) - \delta\mathbf{a}(\delta\mathbf{b}.\nabla)]\,.\mathbf{F}.$$

Using vector identities

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}.\mathbf{a})\,\mathbf{b} - (\mathbf{c}.\mathbf{b})\mathbf{a}$$

and $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \mathbf{b}.(\mathbf{c}\times\mathbf{a}) = \mathbf{c}.(\mathbf{a}\times\mathbf{b})$,

we get

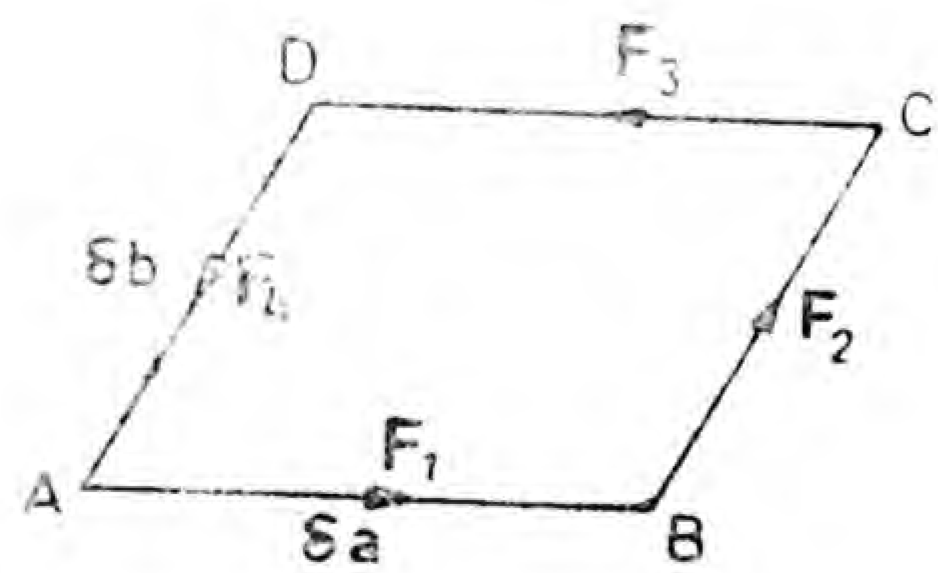


Fig. 1.15.

$$\oint \mathbf{F}.d\mathbf{l} = [(\delta\mathbf{a}\times\delta\mathbf{b})\times\nabla].\mathbf{F}$$

$$= (\delta\mathbf{a}\times\delta\mathbf{b})\,.\,(\nabla\times F). \qquad \ldots(68)$$

Putting $\delta\mathbf{a}\times\delta\mathbf{b} = \delta\mathbf{S} =$ Area of $ABCD$, we get

$$\oint \mathbf{F}.d\mathbf{l} = (\nabla\times\mathbf{F})\,.\,\delta\mathbf{S}. \qquad \ldots(69)$$

Summing these results for all the meshes, we get

$$\oint \mathbf{F}.d\mathbf{l} = \sum_{\text{all the meshes}} \mathbf{F}.d\mathbf{l} = \iint_S (\nabla\times\mathbf{F}).d\mathbf{S}.$$

or $$\oint \mathbf{F}.d\mathbf{l} = \iint_S \hat{n}.\,\text{curl}\,\mathbf{F}\,dS. \qquad ...(70)$$

Hence the Stoke's theorem.

Like the divergence theorem this theorem is very useful in the evaluation of integrals.

1.13. GREEN'S THEOREM

Green's theorem can be derived from Gauss's divergence theorem as follows. Substituting vector field **F** as the product of one scalar ϕ and grad of another scalar ψ, *i.e.*, $\mathbf{F}=\phi\nabla\psi$, we get

$$\nabla.\mathbf{F} = \frac{\partial}{\partial x}\left(\phi\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\phi\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\phi\frac{\partial\psi}{\partial z}\right)$$

$$=\phi\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}+\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}$$

$$=\phi\nabla^2\psi+\nabla\phi.\nabla\psi. \qquad ...(71)$$

Hence Gauss's theorem reduces to

$$\iiint_V \phi\nabla^2\psi+\nabla\phi.\nabla\psi)\,dV = \iint_S (\phi\nabla\psi).d\mathbf{S}. \qquad ...(72)$$

This is known as the first form of Green's theorem. After inter-changing ϕ and ψ, we have

$$\iiint_V (\psi\nabla^2\phi+\nabla\psi.\nabla\phi)\,dV = \iint_S (\psi\nabla\phi).d\mathbf{S} \qquad ...(73)$$

On subtracting Eq. (73) from Eq. (72), we get

$$\iiint_V (\phi\nabla^2\psi-\psi\nabla^2\phi)\,dV = \iint_S (\phi\nabla\psi-\psi\nabla\phi).d\mathbf{S} \qquad ...(74)$$

This is referred to as the *second form of Green's theorem.* These are of extreme importance in the fields of electrodynamics and hydrodynamics.

Some Useful Vector Relations involving the vector ∇

$$(\mathbf{u}.\nabla)\,\phi=\mathbf{u}.(\nabla\phi)$$

$$(\mathbf{u}.\nabla)\,\mathbf{r}=\mathbf{u}.$$

$$\text{div}\,(\mathbf{u}+\mathbf{v})=\nabla.(\mathbf{u}+\mathbf{v})=\nabla.\mathbf{u}+\nabla.\mathbf{v}.$$

$$\text{curl}\,(\mathbf{u}+\mathbf{v})=\nabla\times(\mathbf{u}+\mathbf{v})=\nabla\times\mathbf{u}+\nabla\times\mathbf{v}$$

$$\text{div}\,(\phi\mathbf{u})=\nabla.(\phi\mathbf{u})=\phi\,\text{div}\,\mathbf{u}+\mathbf{u}.\,(\text{grad}\,\phi).$$

$$\text{curl}\,(\phi\mathbf{u})=\nabla\times(\phi\mathbf{u})=\phi\,\text{curl}\,\mathbf{u}+(\text{grad}\,\phi)\times\mathbf{u}.$$

$$\text{div}\,(\mathbf{u}\times\mathbf{v})=\mathbf{v}.\ \text{curl}\ \mathbf{u}-\mathbf{u}.\ \text{curl}\ \mathbf{v}.$$

$$\text{curl}\,(\mathbf{u}\times\mathbf{v})=(\mathbf{v}.\ \text{grad})\ \mathbf{u}-(\mathbf{u}.\ \text{grad})\ \mathbf{v}+\mathbf{u}\,\text{div}\ \mathbf{v}-\mathbf{v}\ \text{div}\ \mathbf{u}.$$

$$\text{grad}\,(\mathbf{u}.\mathbf{v})=\mathbf{u}\times\text{curl}\ \mathbf{v}+\mathbf{v}\times\text{curl}\ \mathbf{u}+(\mathbf{u}.\ \text{grad})\mathbf{v}+(\mathbf{v}.\text{grad})\mathbf{u}.$$

Second order differential operators

$$\text{div curl}\ \mathbf{u}=0$$

$$\text{div}\,(\text{grad}\ \phi)=\nabla.(\nabla\ \phi)=\nabla^2\phi=\partial^2\phi/\partial x^2+\partial^2\phi/\partial^2 y+\partial^2\phi/\partial z^2.$$

$$\text{curl grad}\ \phi=0$$

$$\text{curl curl}\ \mathbf{u}=\text{grad div}\ \mathbf{u}-\nabla^2\mathbf{u}.$$

Exercises

Example 1. *Prove that the four points* $4\mathbf{i}+5\mathbf{j}+\mathbf{k}$, $-(\mathbf{j}+\mathbf{k})$, $3\mathbf{i}+9\mathbf{j}+4\mathbf{k}$ *and* $4(-\mathbf{i}+\mathbf{j}+\mathbf{k})$ *are coplaner.*

Let the given vectors represent the points A, B, C and D respectively, then

$$\mathbf{a}=\mathbf{AB}=\text{Position vector of }\mathbf{B}-\text{Position vector of }\mathbf{A}$$

$$=-(\mathbf{j}+\mathbf{k})-(4\mathbf{i}+5\mathbf{j}+\mathbf{k})=-(4\mathbf{i}+6\mathbf{j}+2\mathbf{k}).$$

Similarly $\mathbf{b}=\mathbf{AC}=3\mathbf{i}+9\mathbf{j}+4\mathbf{k}-(4\mathbf{i}+5\mathbf{j}+\mathbf{k})=-\mathbf{i}+4\mathbf{j}+3\mathbf{k}.$

and $\mathbf{c}=\mathbf{AD}=4(-\mathbf{i}+\mathbf{j}+\mathbf{k})-(4\mathbf{i}+5\mathbf{j}+\mathbf{k})=-8\mathbf{i}-\mathbf{j}+3\mathbf{k}.$

If the points A, B, C and D are coplaner, then vectors **a**, **b** and **c** will also be coplaner. It is possible only when

$$\mathbf{a}\ .\ (\mathbf{b}\times\mathbf{c})=0$$

$$\text{Now }\mathbf{a}.\ (\mathbf{b}\times\mathbf{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=\begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}=0$$

Hence the points A, B, C, D are coplaner.

Example 2. *For a position vector* $\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$, *show that*

(*a*) *div* $\mathbf{r}=3$,

(*b*) *div* $(r^n\,\mathbf{r})=(3+n)\,r^n$,

(*c*) *curl* $\mathbf{r}=0$,

(*d*) *curl* $(\mathbf{r}/r^3)=0$.

(*a*) We know that

$$\text{div}\ \mathbf{F}=\nabla\ .\mathbf{F}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right).\ (F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k})$$

$$\text{div } \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

In the present case $\mathbf{F} = \mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, therefore

$$\text{div } \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

(*b*) In this case $\mathbf{F} = r^n \, \mathbf{r} = r^n (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$,

$$\therefore \text{div } \mathbf{F} = \text{div } (r^n \, \mathbf{r}) = \frac{\partial}{\partial x} (r^n x) + \frac{\partial}{\partial y} (r^n y) + \frac{\partial}{\partial z} (r^n z)$$

$$= r^n + x n r^{n-1} \frac{\partial r}{\partial x} + r^n + y n r^{n-1} \frac{\partial r}{\partial y} + r^n + z n r^{n-1} \frac{\partial r}{\partial z}$$

$$= 3r^n + n r^{n-1} \left[x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z} \right]$$

$$= 3r^n + n r^{n-1} \left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r} \right] = 3r^n + n r^{n-1} . \frac{r^2}{r}$$

$$= (3 + n) \, r^n.$$

(*c*) Since curl of a vector field $\mathbf{F}$ is given as

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k}.$$

In this case $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, hence

$$\text{curl } \mathbf{F} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \mathbf{k}.$$

Since x, y, z are independent co-ordinates, hence

$$\frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = \frac{\partial z}{\partial y} = \frac{\partial y}{\partial z} = \frac{\partial x}{\partial z} = \frac{\partial z}{\partial x} = 0.$$

and $\quad \text{curl } \mathbf{F} = 0\mathbf{i} = 0\mathbf{j} + 0\mathbf{k} = 0.$

(*d*) In this case

$$\mathbf{F} = \frac{\mathbf{r}}{r^3} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r^3}$$

$$\therefore \text{curl } \mathbf{F} = \mathbf{i} \left[\frac{\partial}{\partial y} \left(\frac{z}{r^3} \right) - \frac{\partial}{\partial z} \left(\frac{y}{r^3} \right) \right] + \mathbf{j} \left[\frac{\partial}{\partial z} \left(\frac{x}{r^3} \right) - \frac{\partial}{\partial x} \left(\frac{z}{r^3} \right) \right]$$

$$+ \mathbf{k} \left[\frac{\partial}{\partial x} \left(\frac{y}{r^3} \right) - \frac{\partial}{\partial y} \left(\frac{x}{r^3} \right) \right]$$

$$\text{curl } \mathbf{F}=\mathbf{i}\left[-\frac{3}{r^4}\left(z\frac{\partial r}{\partial y}-y\frac{\partial r}{\partial z}\right)\right]+\mathbf{j}\left[-\frac{3}{r^4}\left(x\frac{\partial r}{\partial z}-z\frac{\partial r}{\partial x}\right)\right]$$
$$+\mathbf{k}\left[-\frac{3}{r^4}\left(y\frac{\partial r}{\partial x}-x\frac{\partial r}{\partial y}\right)\right]$$

Since $r=(x^2+y^2+z^2)^{1/2}$, hence $\partial r/\partial x=x/r$, $\partial r/\partial y=y/r$ and $\partial r/\partial z=z/r$. Substituting these values in the above relation, we have

$$\text{curl } \mathbf{F}=\mathbf{i}\left[-\frac{3}{r^4}\left(\frac{zy}{r}-\frac{yz}{r}\right)\right]+\mathbf{j}\left[-\frac{3}{r^4}\left(\frac{xz}{r}-\frac{zx}{r}\right)\right]$$
$$+\mathbf{k}\left[-\frac{3}{r^4}\left(\frac{yx}{r}-\frac{xy}{r}\right)\right]=0.$$

Example 3. *If* $\mathbf{r}$ *is a position vector and* $\mathbf{a}$ *is a constant vector then prove that*

$$\nabla\,(\mathbf{a}.\mathbf{r})=\mathbf{a}.$$

Using the vector identity

grad $(\mathbf{A}.\mathbf{B})=\mathbf{A}\times\text{curl } \mathbf{B}+\mathbf{B}\times\text{curl } \mathbf{A}+(\mathbf{A}.\text{ grad})\,\mathbf{B}+(\mathbf{B}.\text{ grad})\,\mathbf{A}$,
we have

$$\nabla\,(\mathbf{a}.\mathbf{r})=\mathbf{a}\times(\nabla\times\mathbf{r})+\mathbf{r}\times(\nabla\times\mathbf{a})+(\mathbf{a}.\nabla)\mathbf{r}+(\mathbf{r}.\nabla)\mathbf{a}. \quad ...(i)$$

As $\mathbf{a}$ is a constant vector, hence curl $\mathbf{a}=0$. We know the vector identities curl $\mathbf{r}=0$ and

$$(\mathbf{A}.\nabla)\mathbf{B}=A_x\frac{\partial \mathbf{B}}{\partial x}+A_y\frac{\partial \mathbf{B}}{\partial y}+A_z\frac{\partial \mathbf{B}}{\partial z}. \quad ..(ii)$$

In the present case if we assume $\mathbf{A}=\mathbf{a}$ and $\mathbf{B}=\mathbf{r}$, then we have

$$(\mathbf{a}.\nabla)\mathbf{r}=a_x\frac{\partial \mathbf{r}}{\partial x}+a_y\frac{\partial \mathbf{r}}{\partial y}+a_z\frac{\partial \mathbf{r}}{\partial z}$$
$$=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}=\mathbf{a}.$$

If $\mathbf{A}=\mathbf{r}$ and $\mathbf{B}=\mathbf{a}$ in equation (*ii*), then we have

$$(\mathbf{r}.\nabla)\mathbf{a}=x\frac{\partial \mathbf{a}}{\partial x}+y\frac{\partial \mathbf{a}}{\partial y}+z\frac{\partial \mathbf{a}}{\partial z}$$
$$=0. \text{ (as } \mathbf{a} \text{ is a constant vector)}$$

Substituting these values in Eq. (*i*), we get

$$\nabla(\mathbf{a}.\mathbf{r})=0+0+\mathbf{a}+0=\mathbf{a}.$$

Hence the result.

Example 4. (*a*) *Calculate grad* $\phi(\mathbf{r})$. (*b*) *Prove that grad is* ⊥ *to the surface for which* ϕ *is constant.*

(*a*) grad $\phi=\mathbf{i}\,\partial\phi/\partial x+\mathbf{j}\,\partial\phi/\partial y+\mathbf{k}\,\partial\phi/\partial z$,

where $\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial r}.\frac{\partial r}{\partial x}=\phi'\frac{\partial r}{\partial x}$, $\frac{\partial\phi}{\partial y}=\phi'\frac{\partial r}{\partial y}$ and $\frac{\partial\phi}{\partial z}=\phi'\frac{\partial r}{\partial z}$.

As $r=(x^2+y^2+z^2)^{1/2}$, hence $\partial r/\partial x=2x/2(x^2+y^2+z^2)^{1/2}=x/r$, $\partial r/\partial y=y/r$ and $\partial r/\partial z=z/r$.

$$\therefore \quad \text{grad } \phi(r)=\mathbf{i}\,\phi'\frac{x}{r}+\mathbf{j}\phi'\frac{y}{r}+\mathbf{k}\phi'\frac{z}{r}$$

$$=\phi'\frac{\mathbf{r}}{r}=\frac{\partial\phi}{\partial r}\frac{\mathbf{r}}{r}=\frac{\partial\phi}{\partial r}\hat{r}$$

$$(b)\quad d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=(\text{grad }\phi).\,d\mathbf{r}.$$

Let $d\mathbf{r}$ lies along the surface for which ϕ=const, then obviously $d\phi=0$ and (grad ϕ). $d\mathbf{r}=0$. It shows that grad ϕ and $d\mathbf{r}$ are mutually $\perp$ to each other for a surface ϕ=const.

Example 5. *Prove the vector identities :*

(*i*) *div curl* $\mathbf{F}=0$. (*ii*) *curl grad* $\phi=0$.

$$(i)\ \text{div curl }\mathbf{F}=\frac{\partial}{\partial x}(\text{curl }\mathbf{F})_x+\frac{\partial}{\partial y}(\text{curl }\mathbf{F})_y+\frac{\partial}{\partial z}(\text{curl }\mathbf{F})_z$$

$$=\frac{\partial}{\partial x}\left[\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right]+\frac{\partial}{\partial y}\left[\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right]+\frac{\partial}{\partial z}\left[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right]$$

$$=\frac{\partial^2 F_z}{\partial x\partial y}-\frac{\partial^2 F_y}{\partial x\partial z}+\frac{\partial^2 F_x}{\partial y\partial z}-\frac{\partial^2 F_z}{\partial y\partial x}+\frac{\partial^2 F_y}{\partial z\partial x}-\frac{\partial^2 F_x}{\partial z\partial y}$$

$$=0.$$

It means that *vector field whose divergence is everywhere zero can be expressed as the curl of some other suitable vector field. The fields whose divergence is zero have their field lines always forming closed curves or the vector field is solenoidal.*

$$(ii)\ (\text{curl grad }\phi)_x=\left[\frac{\partial}{\partial y}(\text{grad }\phi)_z-\frac{\partial}{\partial z}(\text{grad }\phi)_y\right]\mathbf{i}$$

$$=\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\mathbf{i}=0.$$

Similarly we can prove that y-and z-components are also zero.

$\therefore$ curl grad$\phi=0$.

It means that *a vector field whose curl is everywhere zero can be expressed as the gradient af another suitable scalar field. The field is known as non-curl, irrotational, or scalar potential field. All the conservative fields are of this type.*

Example 6. *A rigid body is rotating with constant angular velocity* ω *about a fixed axis, if* $\mathbf{v}$ *is the velocity of a point of the body, prove that curl* $\mathbf{v}=\omega$. *Give its physical significance.*

Consider a rigid body which is rotating with an angular velocity ω about an axis OA, where O is a fixed point in the body. Then any point P on the body moves in a circular path about OA with a tangential linear velocity $\mathbf{v}$ $(=\omega\times\mathbf{r})$, where $\mathbf{r}$ is the distance of point P from the fixed point O. Since the angular velocity ω is a vector constant for all points on the body and can be written as

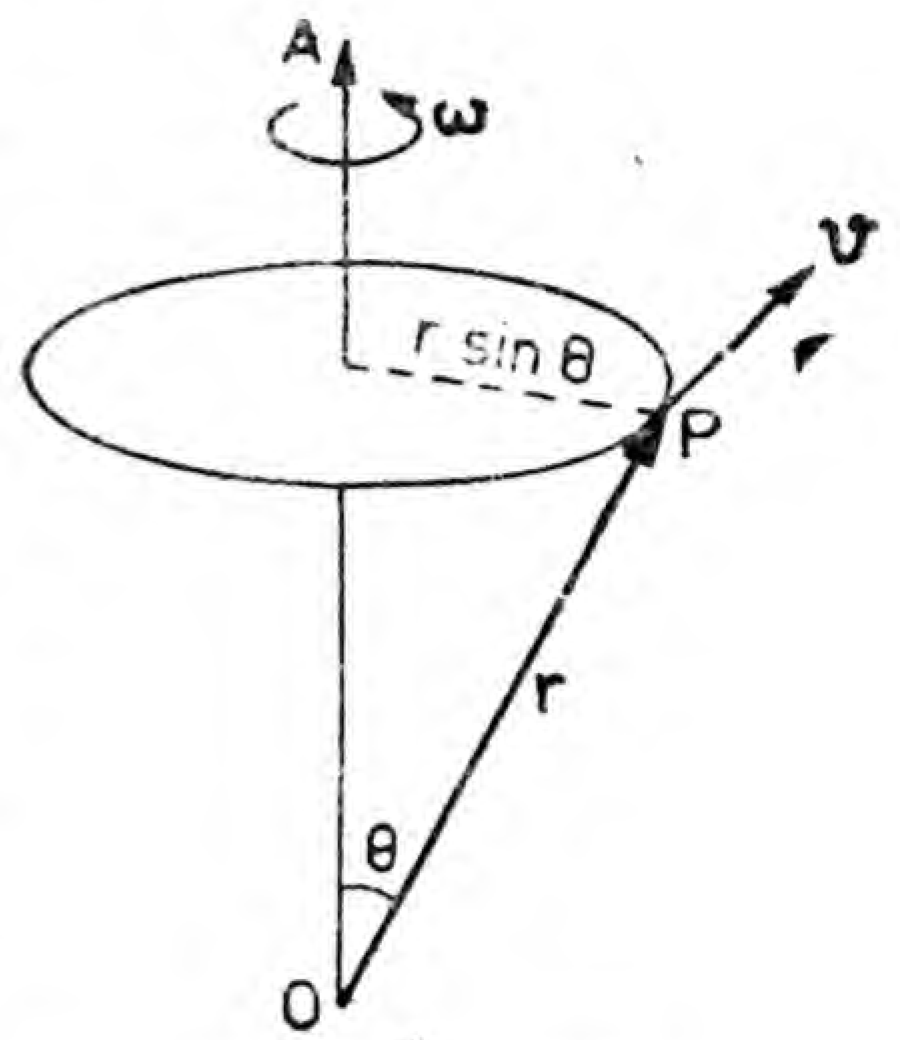


Fig. 1.16.

$\omega=\omega_x\mathbf{i}+\omega_y\mathbf{j}+\omega_z\mathbf{k}$, hence the components ω_x, ω_y, ω_z are independent of the co-ordinates $\mathbf{r}$ of the point.

$$\therefore\quad \mathbf{v}=\omega\times\mathbf{r}=(\omega_y z-\omega_z y)\mathbf{i}+(\omega_z x-\omega_x z)\,\mathbf{j}+(\omega_x y-\omega_y x)\,\mathbf{k}$$

and $\nabla\times\mathbf{v}=2\omega_x\mathbf{i}+2\omega_y\mathbf{j}+2\omega_z\mathbf{k}=2\omega$.

Hence when the rigid body is in motion the curl of its linear velocity at any point is equal to twice its angular velocity.

Thus we see that if a motion of a rigid body is such that the velocity has a curl, the ultimate particles of the body are in rotation with an angular velecity, which is equal to half of that curl. The motion is described as *rotational* or *vortical.*

Example 7. *For a position vector* $\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$, *find the values of* (*i*) *grad* $(1/\mathbf{r})$ (*ii*) *and grad* r^m.

$$\text{grad}\ \frac{1}{r}=\nabla\left(\frac{1}{r}\right)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$=\mathbf{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}+\mathbf{j}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y}+\mathbf{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$=-\frac{1}{r^2}\left[\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}\right]$$

Since $\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$, hence $r^2=\mathbf{r}.\mathbf{r}=x^2+y^2+z^2$

and $\partial r/\partial x=x/r,\ \partial r/\partial y=y/r,\ \partial r/\partial z=z/r.$

Substituting these values in the above relation, we get

$$\text{grad}\left(\frac{1}{r}\right)=-\frac{1}{r^2}\left[\mathbf{i}\frac{x}{r}+\mathbf{j}\frac{y}{r}+\mathbf{k}\frac{z}{r}\right]$$

$$\text{grad}\left(\frac{1}{r}\right)=-\frac{1}{r^3}\left[\mathbf{i}x+\mathbf{j}y+\mathbf{k}z\right]=-\frac{\mathbf{r}}{r^3}.$$

Similarly

$$\text{grad } r^m=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(r^m)$$

$$=mr^{m-1}\left[\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}\right]$$

$$=mr^{m-1}\left[\mathbf{i}\frac{x}{r}+\mathbf{j}\frac{y}{r}+\mathbf{k}\frac{z}{r}\right]$$

$$=mr^{m-2}\left[\mathbf{i}x+\mathbf{j}y+\mathbf{k}z\right]$$

$$=mr^{m-2}\,\mathbf{r}.$$

Example 8. Prove that curl curl $\mathbf{A}$=grad div $\mathbf{A}-\nabla^2\mathbf{A}$.

We know that

$$\text{curl }\mathbf{A}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{k}.$$

and $\nabla=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)$. Therefore

$$\text{curl curl }\mathbf{A}=\nabla\times\nabla\times\mathbf{A}$$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\left[\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_y}{\partial x}\frac{\partial A_x}{\partial y}\right)\mathbf{k}\right].$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{k}-\frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{j}$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{k}+\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{i}$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{j}-\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{i}$$

Let us consider only the x component and call it curl_x curl **A**.

$$\text{curl}_x \text{ curl } \mathbf{A} = \left[\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] i$$

$$= \left[\frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y \partial y} - \frac{\partial^2 A_x}{\partial z \partial z} + \frac{\partial^2 A_z}{\partial z \partial x}\right] \mathbf{i}.$$

Adding and subtracting $\partial^2 A_x/\partial x^2$ in this expression, we get

$$\text{curl}_x \text{ curl } \mathbf{A} = \left[\frac{\partial^2 A_x}{\partial x \partial x} + \frac{\partial^2 A_y}{\partial y \partial x} + \frac{\partial^2 A_z}{\partial z \partial y} - \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\right] \mathbf{i}$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A_x\right] \mathbf{i}$$

$$= \mathbf{i} \frac{\partial}{\partial x} (\text{div } \mathbf{A}) - \nabla^2 (A_x \mathbf{i})$$

$$= \text{grad}_x (\text{div } \mathbf{A}) - \nabla^2 (A_x \mathbf{i})$$

Similarly, we have

$$\text{curl}_y \text{ curl } \mathbf{A} = \text{grad}_y (\text{div } \mathbf{A}) - \nabla^2 (A_y \mathbf{j})$$

and $$\text{curl}_z \text{ curl } \mathbf{A} = \text{grad}_z (\text{div } \mathbf{A}) - \nabla^2 (A_z \mathbf{k})$$

Therefore the summation gives

$$\text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}.$$

This relation can also be derived by using the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \, (\mathbf{a}.\mathbf{c}) - (\mathbf{a}.\mathbf{b}) \, \mathbf{c}.$$

or $$\nabla \times \nabla \times A = \nabla \, (\nabla.\mathbf{A}) - (\nabla.\nabla) \, \mathbf{A}$$

$$= \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}.$$

Oral Questions—

1. Show how three equal magnitude vectors would have to be oriented if they were to add to give zero. Can this be done with three unequal vectors ? Two unequal vectors ?

2. Find the dot and cross products of two similar vectors ?

3. Why is the work a scalar quantity and the moment of the force a vector quantity ?

4. If ∇ operates v when u is assumed as constant, then which of the following is the correct form

$$(\nabla \, . \, \mathbf{u})\mathbf{v} \text{ or } (\mathbf{u} \, . \, \nabla)\mathbf{v} \, ?$$

5. Find the surface integral of *a* curl of *a* vector field when the surface is closed.

6. If two vectors **A** and **B** are not parallel in question $a\mathbf{A}+b\mathbf{B}=0$, then find the values of a and b.

Problems—

1. Prove that the line joining the vertex of a parallelogram and the mid-point of the opposite side trisects the diagonal.

2. Prove that $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$, $\mathbf{a}\times\mathbf{a}=0$ and $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=2(\mathbf{a}\times\mathbf{b})$.

3. If **A** and **B** are the sides of a parallelogram and **C** and **D** are the diagonals, show that $C^2+D^2=2(A^2+B^2)$.

4. Prove that the components of **F** in the plane of **a** and **F**, along and perpendicular to **a** are $\dfrac{\mathbf{a}\cdot\mathbf{F}}{a^2}\,\mathbf{a}$ and $\dfrac{(\mathbf{a}\times\mathbf{F}\times\mathbf{a})}{a^2}$ respectively

5. Find grad v, where $v=r^m$ and $r=(x^2+y^2+z^2)^{1/2}$

[mr^{m-2}]

6. Find div **F** and curl **F**, where $\mathbf{F}=\text{grad}\,(x^3+y^3+z^3-3xyx)$.

[$6(x+y+z)$ and 0]

7. Evaluate $\nabla^2 r$ where $r=(x^2+y^2+z^2)^{1/2}$.

8. Show that the potential function $\phi=q(x^2+y^2+z^2)^{-1/2}$ satisfies the Laplace's equation.

2

Electric Charge and Electrostatic Field

2.1. ELECTRIC CHARGE

It is a well established fact that bodies which are not in contact can interact with each other without the use of an intervening medium. The best known interaction of this kind is *gravitation*. In certain circumstances the interaction between separated bodies can be such that the forces produced on them completely overshadow the gravitational attraction. In these cases we say that these bodies are magnetic or charged. We can show that a glass rod rubbed with silk will repel the second glass rod rubbed with silk and will attract a hard rubber rod rubbed with fur. Two hard rubber rods rubbed with fur will repel each other. We explain these facts by saying that rubbing a rod gives it an electric charge and the charges on the glass rod and on the hard rubber must be different in nature. It is also clear that *like charges repel and unlike charges attract*. Benjamin Franklin, American Physicist, named the kind of charge that appears on the glass positive and the kind that appears on the hard rubber negative. He also observed that equal negative and positive charges are obtained at the same time by the rubber on the one hand and by the body rubbed on the other. Electric effects are not limited to glass or rubber, any substance rubbed with any other under suitable conditions will become charged. Now-a-days it is assumed that the matter contains equal amount of negative (electrons) and positive (protons) electricity. In the process acquiring the electric charge, a small amount of one kind of electricity is transferred from one to the other body. In this way one body would become positive and other negative.

Charge is quantized. The atomic theory of matter has shown that fluids (*e.g.*, water and air) are not continuous, but are made up of atoms. Milikan's oil drop experiment shows that there is the exact equality in the charges carried by all charged drops. The charge is written as ne, where n is an integer number. The symbol 'e' is known as electronic charge. It is also observed that electron and proton have charges equal in magnitude but opposite in sign. No one has been able to detect a charge smaller in magnitude than the charge of the electron *e*. In addition the magnitudes of all other charges are found to be integer multiples of the magnitude of the charge on the electron. Thus we say that charge exists in discrete packets rather than in continuous amounts and hence is said to be *quantized*. The basic packet, or quantum, of charge has magnitude *e*. All charged elementary particles, known upto this time carry charges of precisely the same magnitude and have their anti-particles. It shows that the *quantization of charge is a deep and universal law of nature*. There are two kinds of charge quanta. Both have the same magnitude. One is that on the electron and other is that on the proton. Higher theories of quantum mechanics are required to explain the quantization of charge.

The quantum of charge *e* is so small that the *graininess* of electricity does not show up in most of the experiments. For example in an ordinary 220 volts 60 watt light bulb, 1.7×10^{18} elementary charges (electrons) enter and leave filament every second.

Conservation of charge. We know that the total charge in an *isolated system* remains constant. By isolated we mean that no matter is allowed to enter or leave the system. The charges can be neither created nor destroyed. An equal quantity of each sign positive and negative is simultaneously produced or disappeared. This is known an *conservation of charge* and is valid both for large scale events and at the atomic and nuclear level, no exceptions have been found. An interesting example of charge conservation is the creation of pair of electron and positron when high energy photons enter a thin walled box in vacuum. Experimentally it was observed that the electric charges of electron and positron were equal in magnitude and opposite in sign. These particles are related to one another as *particle* to *antiparticle*. The reverse process, in which energy appears in the form of two gamma rays when an electron and a positron are brought close to each other, is also observed. This process is known as *annihilation* process. In these processes the net charge is zero both before and after the event so that *charge is conserved*.

Another example of the conservation of charge is found in *radioactive decay*. Few decay processes are :

$$_{92}U^{238} \rightarrow {}_{90}Th^{234} + {}_{2}He^{4}$$
$$_{88}Ra^{226} \rightarrow {}_{86}Rn^{222} + {}_{2}He^{4}.$$

In these processes the totol amount of charge present before disintegration (92 and 88 for the first and second processes respectively) is the same as that present after the disintegration.

Charge conservation is also valid in *nuclear reactions*. Few examples are :

$$_{20}Ca^{44}+{}_{1}H^{1}\rightarrow{}_{21}Sc^{44}+{}_{0}n^{1}$$
$$_{2}He^{4}+{}_{7}N^{14}\rightarrow{}_{8}O^{17}+{}_{1}H^{1}$$
$$_{3}Li^{7}+{}_{1}H^{1}\rightarrow 2\,{}_{2}He^{4}.$$

Here we see that the sum of the atomic numbers (*i.e.*, no. of protons in the atom) before the reaction is exactly equal to the sum of the atomic numbers after the reaction. It means that total charge remains constant or *charge is conserved.*

The law of conservation of charge is also true for relativistic motion. In other words we can say that the total electric charge of an isolated system is *relativistically invariant*.

Unit of charge. One cannot explain the charge in terms of any other known property. In macroscopic charging process the number of electrons involved is very large and we use a unit of charge, the *coulomb*. The *coulomb* is defined as the amount of charge that flows through a wire per second if there is a steady current of one ampere in the wire. In nuclear or atomic problems or in microscopic processes, the unit of charge is taken as the charge on the electron, *i.e.*, *electronic charge*, *e*, which is equal to 1.6021×10^{-19} coulomb.

Conductors and Insulators. For the purpose of electrostatic theory all substances can be divided into two main classes : *conductors and insulators*. In conductors electric charges are free to move from one place to another, whereas in insulators they are tightly bound to their respective atoms. In an uncharged body there are an equal number of positive and negative charges.

The examples of conductors of electricity are the metals, human body and the earth and that of insulators are glass, hard rubber and plastics. In metals, the free charges are free electrons, known as conduction, metallic or free electrons. But in electrolytes, each molecule of electrolyte separates into+vely and—vely charged parts which can move independently of each other. Although there is no perfect insulator or perfect conductor. The insulating ability of fused quartz is about 10^{25} times as great as that of copper and hence used as perfect insulator. The concepts of perfect insulator and of perfect conductor are useful in electrostatic problems. There are a number of substances that are neither good conductors of electricity nor good insulators. These substances are called *semiconductors*.

Charge and Matter. Every matter is consisted of neutral atoms. The atoms are made up of a dense+vely charged nucleus surrounded by the electrons in the orbital motion. The radius of the nucleus is of the order of fermi ($1\,f=10^{-15}m$) and is about 10^{-5} times smaller than the radius of the outer electron or that of an atom. In the nucleus protons and neutrons are packed by the strong *attractive forces,* known as *nuclear forces.* These forces are much stronger than the electrostatic repulsive forces between the protons.

The forces that bind the electrons of an atom to the nucleus, the forces that bind atoms together to form molecules and the forces that bind molecules together to form matter may be described with the help of Coulomb's law and the structure of the matter.

2.2. COULOMB'S LAW

Charles Augustin de Coulomb performed following experiment in 1785 to find how the force varies with the distance between two point charges. In his experiment, two light balls were fixed at the ends of a light insulating rod and the whole system was suspended by means of a torsion fibre at its centre of mass (Fig 2.1). One ball *a* was given a charge and a third charged ball c supported on the insulating handle was brought near the ball *a*. The charge on *a* was repelled and the rod was rotated and the fibre was twisted. The suspension head was rotated to restore the original position. This experiment was repeated for different distances between balls *a* and *c* and corresponding angles through which the suspension head was rotated to restore the original position were observed. From these results, *Coulomb* gave a law, can be stated as :

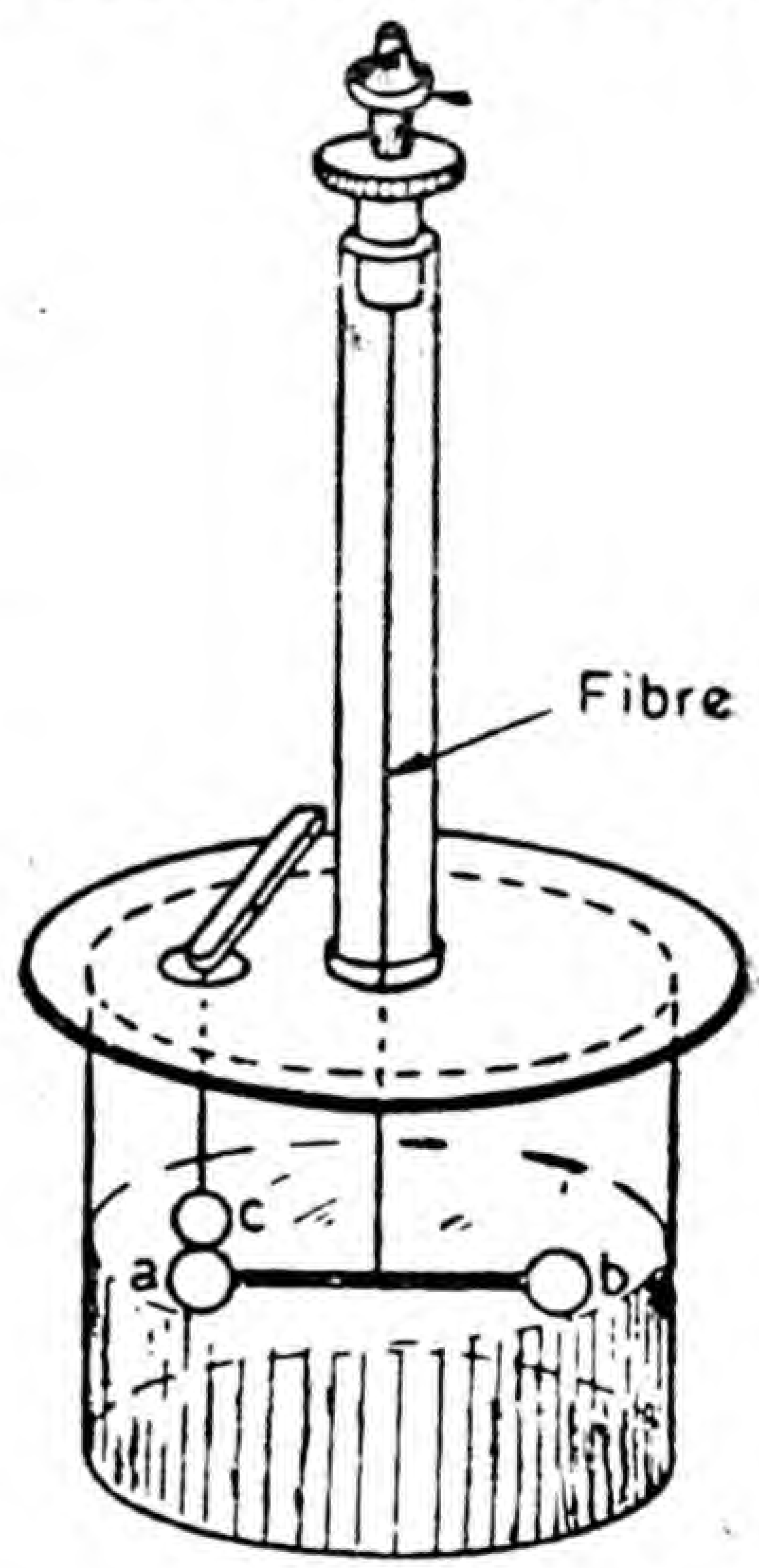


Fig. 2.*1*.

(*a*) The force which two charged bodies 1 and 2 (say), whose dimensions are small compared to their separation (*i.e.*, point charges). exert on one another has a direction of the line joining the charges and is inversely proportional to the square of their separation ***r***. *i.e.*,

$$\mathbf{F} \propto \frac{1}{r^2}\hat{r}, \qquad (i)$$

where $\hat{r}=\mathbf{r}/r$ is the unit vector in the direction from charge 1 to charge 2 if the force **F** is acting on charge 2 due to charge 1 (Fig 2.2).

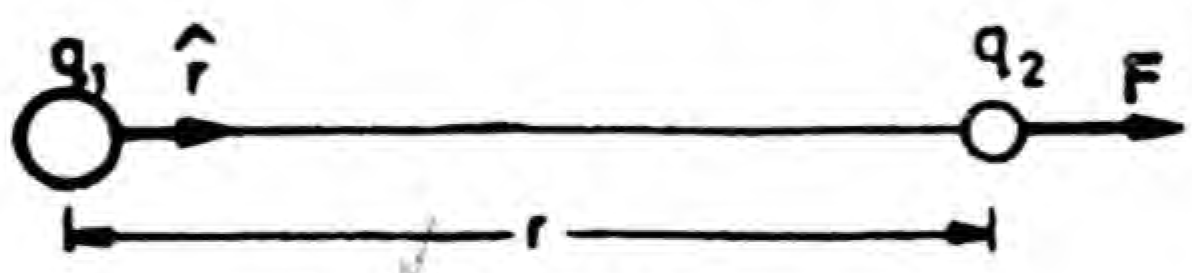


Fig. 2.2.

(*b*) This force is proportional to the product of the charges on these charged bodies.

$$F \propto q_1 q_2. \qquad (ii)$$

(*c*) This force acts as a repulsive force when q_1 and q_2 have same sign and as an attractive force when two signs are opposite.

(*d*) The force between any two charges is independent of the presence of other charges.

Combinning these relations we have

$$F \propto \frac{q_1 q_2}{r^2} \hat{r}. \qquad ...(1)$$

It is better to write $\mathbf{F}_2$ instead of **F** as we are concerned with the force on charge 2 due to charge 1. Hence the force $\mathbf{F}_1$ on charge 1 due to charge 2 will be $-\mathbf{F}_2$.

In Gaussian system of units the unit of charge is thus defined so that upon a quantity of electricity equal to itself, at a distance 1 cm., it exerts a force of 1 dyne. In this case constant of proportionality for the charges in vacuum or air becomes unity. In SI units the constant of proportionality is usually written in a more complex way as $1/4\pi\varepsilon_0$. Thus we have

$$\mathbf{F}_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}. \qquad (2)$$

There is no choice about the constant ε_0, *known as the permittivity of free space*. It must have that value which makes the right hand side equal to the left hand side. Thus we have

$$\varepsilon_0 = 8.85418 \times 10^{-12} \text{ coul}^2/\text{nt.m}^2 \text{ or } 1/4\pi\varepsilon_0 = 9 \times 10^9 \text{ nt.m}^2/\text{coul}^2.$$

If more than two charges are present, the force exerted on any one say q_1, by all the others q_2, q_3, q_4, etc., can be given by adding vectorially all the forces obtained by using Eq. (2).

$$\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \cdots, \qquad (3)$$

where $\mathbf{F}_{12}$ stands for the force exerted on q_1 by q_2. Thus we have

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} + \frac{q_1 q_3}{r_{13}^2} \hat{r}_{13} + \cdots \right] \qquad ...(4)$$

In the Gaussian system of units the constant of proportionality in Coulomb's law is arbitrarily chosen equal to unity and the units of charge, current, etc. thereby established. In SI units we start with the definition of the unit or current strength and from this the units of the other quantities such as voltage, charge, etc. were deduced.

Here the unit vectors $\hat{r}_{12}$ and $\hat{r}_{13}$ have the directions of the lines from q_2 to q_1 and q_3 to q_1 respectively.

For the force on q_1 due to a number of charges, we may write

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_1 q_i}{r_{1i}^2} \hat{r}_{1i}. \qquad ...(5)$$

This is just the *superposition principle* for forces.

Let us consider situations where the charge is spread over a region instead of being concentrated at particular points. To calculate the total force on q_1, let us consider the charge as made up of small charge elements and then the forces due to each element dq is added vectorially. As the charge distribution is continuous, hence the summation is replaced by integration. Thus we have

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \int q_1 \frac{dq}{r^2} \hat{r} = \frac{q_1}{4\pi\varepsilon_0} \int \frac{dq}{r^2} \hat{r}, \qquad ...(6)$$

where $\hat{r}$ is a variable unit veotor that points from each charge element dq toward the location of charge q. If the charge is distributed over a volume dv, this equation may be written as

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \int q_1 \frac{q\,dv}{r^2} \hat{r} \qquad ...(7)$$

where q is the charge density.

Coulomb's law applies to point charges. In the macroscopic sense a point charge is one whose spatial dimensions are very small compared with any other length used in the problem. It also applies to the interactions of elementary particles, such as electrons and protons. It is found to hold even for electrostatic repulsion between protons inside the nucleus, however nuclear forces dominate over this repulsion. The spontaneous emission of α-particles, breaking up of the nucleus into two large fragments, presence of more neutrons than the protons in the heavy nuclei are the Coulomb's repulsion effects. We do not know whether this law holds for very large astronomical distances or not.

Coulomb's law has the same form as Newton's law of gravitation. The analogy between these laws is given below :

Coulomb's Law	Newton's gravitational law
(1) It is based on experimental observations.	It is based on speculations concerning the fall of an apple towards earth.

(2) Force is directly proportional to the product of charges.	Force is directly proportional to the product of masses.
(3) Force varies inversely as the square of the distance between two charges.	Force varies inversely as the square of the distance between the masses.
(4) Force between the charges $F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$.	Force between two masses $F = G\frac{m_1 m_2}{r^2}$.
(5) Force is repulsive when charges are of similar nature and is attractive when they are unlike.	Force between two masses is always attractive.
(6) Force between any two charges is independent of the presence of other charges.	Force between any two masses is independent of the presence of other masses.
(7) ϵ_0 is constant for a free space.	G is a universal constant.
(8) ϵ_0 depends upon units of force, charge and distance.	G also depends upon units of force, mass and length.

2.3 ELECTRIC FIELD

The gravitational force can be explained by assuming that every point in space near the earth is associated with a field, known as *gravitational field.* Similarly the space surrounding a charged body is associated with a field known as *electric field.*

If a charge q_1 is placed at any point, it sets up an electric field in the space around itself. This field is indicated by the shaded region in Fig. 2.3. If a charge q_2 is placed in the field region of charge q_1, the former will experience a force **F**.

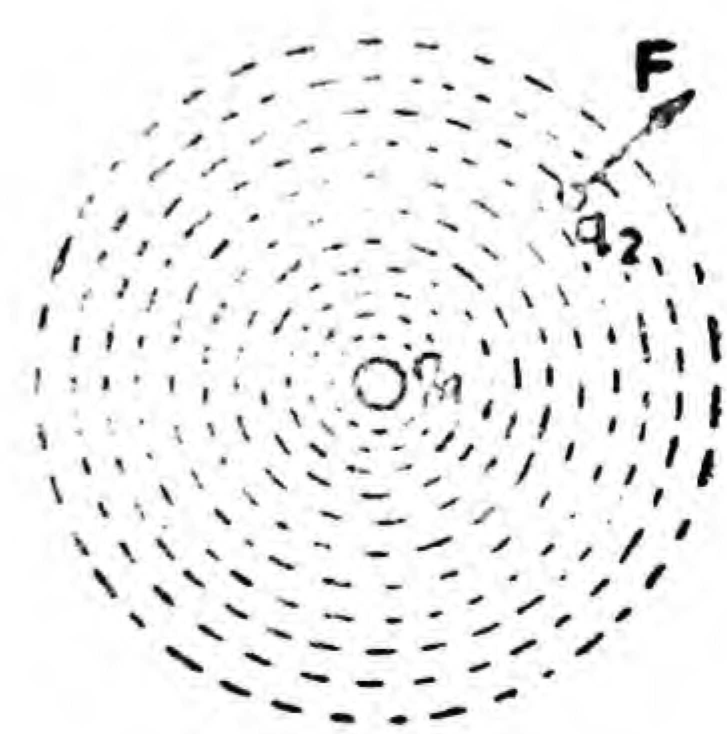


Fig. 2.3.

In this way the field plays an intermediary role in the forces between the charges.

Electric Field Strength. A test charge q_0 (assumed positive for convenience) is placed at any point in the region of any charge where we want to calculate the electric field strength. If this test charge experiences a force **F**, then the electric field intensity or strength **E** at that point is defined as

$$\mathbf{E}=\mathbf{F}/q_0.$$

The direction of **E** is the direction of **F** and its unit is newton/coulomb. We must assume a test charge as small as possible so that it will not influence the behaviour of the primary charges that are responsible for the field to be determined. Thus Eq. (8) should be replaced by

$$\mathbf{E}=\underset{q_0\to 0}{\text{Limit}}\ \frac{\mathbf{F}}{q_0}. \qquad ...(9)$$

Actually the *test charge is fictitious*. We merely ask what would be the force on it, if placed at the observation point. The requirement that the test charge be vanishingly small compared with all sources of the field limits the practical validity of the definition, *i.e.* the definition is suitable for macroscopic phenomena only.

The force experienced by test charge q_0 placed at a distance r from a point charge q can be written at once from Coulomb's law as

$$\mathbf{F}=\frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\ \hat{r}.$$

$$\therefore\ \mathbf{E}=\underset{q_0\to 0}{\text{Limit}}\ \frac{\mathbf{F}}{q_0}=\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}.\ =-\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r}\right). \qquad ...(10)$$

Its direction is radial from q pointing outward if q was+ve. It is inward if q was−ve.

If we consider co-ordinates of the points instead of absolute distance between the charges, then Eq. (10) will become

$$\mathbf{E}(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\frac{q_1\,(\mathbf{x}-\mathbf{x}_1)}{|\,\mathbf{x}-\mathbf{x}_1\,|^3}, \qquad ...(11)$$

where electric field is calculated at point **x** due to a point charge q_1 placed at point $\mathbf{x}_1$ and $|\,\mathbf{x}-\mathbf{x}_1\,|$ is the absolute distance between these points.

2.4. CALCULATION OF ELECTRIC FIELD STRENGTH

To find **E** for a group of point charges, we first calculate $\mathbf{E}_i$ due to each charge at the given point as if it was the only charge present and then add them vectorially.

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_3+...\ =\sum_{i=1}^{n}\mathbf{E}_i=\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_{0i}^2}\ \hat{r}_{0i},$$

where $\hat{r}_{0i}$ is the vector from the i^{th} charge in the system to the point in space for which **E** is being calculated.

If the charge distribution is continuous, the field strength can be computed by dividing the charge into infinitesimal elements dq and integrating the field contributions due to all the charge elements.

$$\mathbf{E}=\int d\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^2}\hat{r}, \qquad \ldots(12)$$

where $\hat{r}$ is the unit vector pointing from dq toward the point in space for which **E** is being calculated.

(*a*) **Electric Dipole.** If two electric charges, equal in magnitude but opposite in signs coincide at any point the electric field strength due to these charges in the space around them vanishes. If the charges suffer a small relative displacement, there is an electric field of sensible magnitude in the neighbourhood of these charges. Such a combination of two charges is termed as *electric dipole.*

The electric field strength **E** at a point Q at a distance r_2 along the perpendicular bisector of the line joining the charges of magnitude q placed at a distance $2a$ apart is given by the relation

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2. \qquad \ldots(13)$$

From Eq. (10)

$$E_1=E_2=\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{a^2+r_2^2}.$$

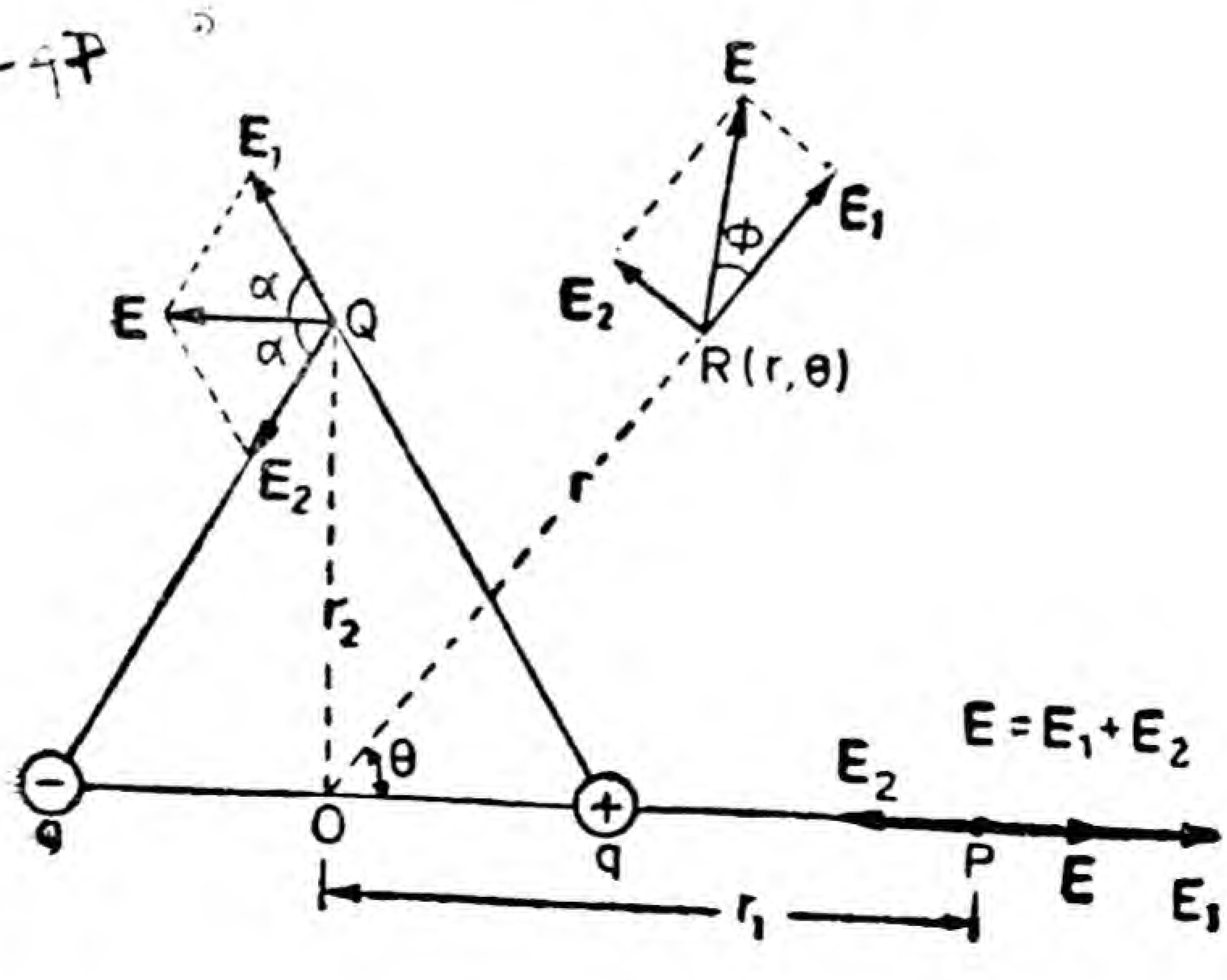


Fig. 2.4.

The vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$ points parallel to the line joining $+q$ and $-q$ and has the magnitude

$$E=2E_1 \cos \alpha$$

$$=2\times\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2+r_2^2}\times\frac{a}{\sqrt{a^2+r_2^2}}=\frac{2aq}{4\pi\varepsilon_0(a^2+r_2^2)^{3/2}}. \quad ...(14)$$

If $r_2 \gg a$, we can neglect a in the denominator and the above equation thus reduces to

$$E=\frac{1}{4\pi\varepsilon_0}\ \frac{(2a)(q)}{r_2^3}=\frac{1}{4\pi\varepsilon_0}\frac{p}{r_2^3}, \quad ...(15)$$

where $p=2aq$, the product of the magnitude of the charge q and the separation $2a$ between the charges and is known as the *electric dipole moment.* Thus we see that the electric field strength due to dipole at distant points along the perpendicular bisector varies as $1/r^3$. whereas for a point charge it varies as $1/r^2$ only. Its direction is opposite to that of dipole moment which is +ve in the direction from −ve charge to +ve charge.

The electric field strength **E** at a point P at a distance r_1 from the centre of the dipole along its axis is given by

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2.$$

The resultant vector **E** points parallel to $\mathbf{E}_1$ and antiparallel to $\mathbf{E}_2$, as $\mathbf{E}_2$ is smaller due to the greater distance of $-q$, *i.e.* along the axis of the dipole. The magnitude

$$E=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(r_1-a)^2}-\frac{q}{(r_1+a)^2}\right]=\frac{q}{4\pi\varepsilon_0}\cdot\frac{4ar_1}{(r_1^2-a^2)^2}. \quad ...(16)$$

As in a dipole the distance between two point charges is negligible in comparison to the distance r_1, *i.e.*, $a \ll r_1$, hence

$$E=\frac{2q(2a)}{4\pi\varepsilon_0 r_1^3}=\frac{1}{4\pi\varepsilon_0}\frac{2p}{r_1^3}. \quad ...(17)$$

Its direction is along $\mathbf{E}_1$, *i.e.*, along the direction of dipole moment. It is also inversely proportional to r_1^3 and proportional to the dipole moment p.

For the calculation of electric field strength E at point R having polar co-ordinates r and θ, let us resolve dipole moment **p** into two components, one parallel and other perpendicular to OR, the line joining point under consideration to the centre of dipole. The field E_1 due to the component $p \cos \theta$ is equal to $2\,p \cos \theta/4\pi\varepsilon_0 r^3$ and the field E_2 due to the component $p \sin\theta$ is equal to $p \sin\theta/4\pi\varepsilon_0 r^3$ which is perpendicular to the former. Hence magnitude of the resultant field **E** is given by

$$E=\frac{p}{4\pi\varepsilon_0 r^3}\sqrt{4 \cos^2\theta+\sin^2\theta}=\frac{p}{4\pi\varepsilon_0 r^3}\sqrt{1+3 \cos^2\theta} \quad ... (18)$$

Its inclination ϕ to OR is given by

$$\tan\phi=\frac{p\sin\theta/4\pi\varepsilon_0 r^3}{2p\cos\theta/4\pi\varepsilon_0 r^3}=\tfrac{1}{2}\tan\theta. \qquad ...(19)$$

(*b*) **Charged Rod.** Let us consider a uniformly charged rod whose linear charge density λ (charge per unit length) is constant. We wish to find the electric field at point P due to this rod. *Coulomb's law cannot be applied directly in this case as it applies only to point charges.* However we can imagine the rod to be divided into a large number of small segments, so that each segment will act as a point charge. Coulomb's law can be applied to this segment. If Δx be the length of the segment at a distance x from point P,

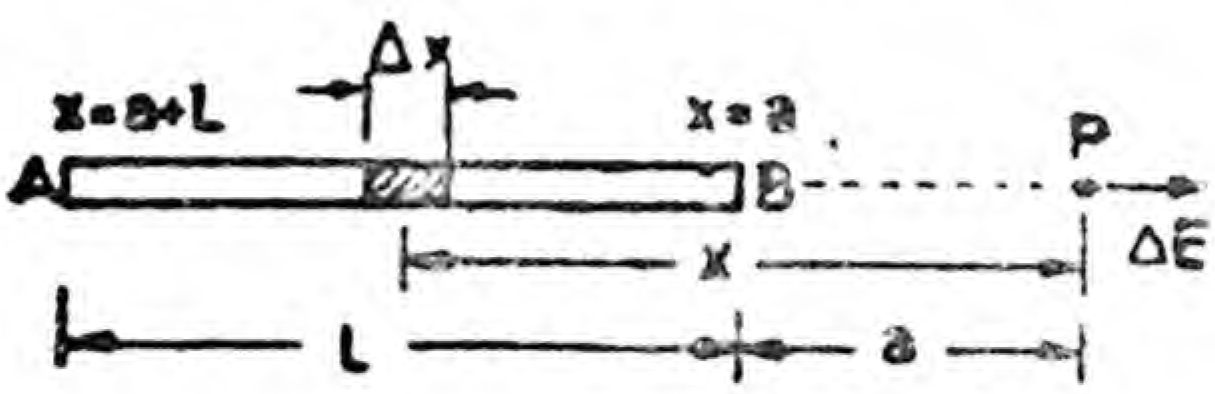


Fig. 2.5.

which is along the length of this rod (Fig 2.5). The electric field due to this segment at point P is given by

$$\Delta\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\,\frac{\lambda\Delta x}{x^2}\hat{x}.$$

It is along **BP** direction. Since the fields due to all segments of the rod are in the same direction as Δ **E**, hence

$$E=\sum\Delta E=\sum\frac{\lambda}{4\pi\epsilon_0}\,\frac{\Delta x}{x^2}. \qquad ...(20)$$

If Δx is sufficiently small, we can replace the sum by an integral, thus we get

$$E=\int_{-(a+L)}^{-a}\frac{\lambda}{4\pi\varepsilon_0}\,\frac{dx}{x^2}=\frac{\lambda}{4\pi\varepsilon_0}\int_{-(a+L)}^{-a}\frac{dx}{x^2}$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{L+a}\right] \qquad ...(21)$$

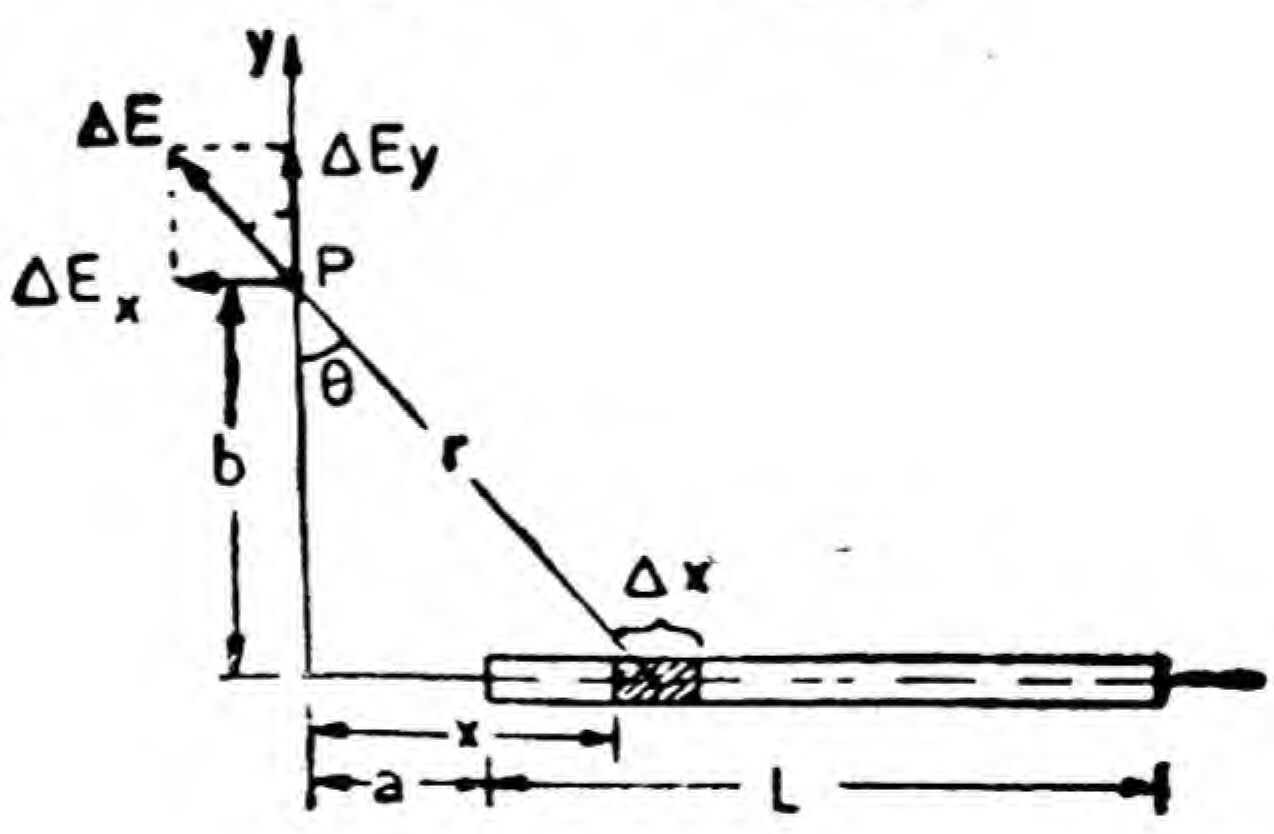


Fig. 2.6.

To find the electric field at a point to one side of the rod we can not write relation (20), as $\triangle E$ due to each segment of the rod will point in a different direction (Fig 2.6). The y—component

$$\triangle E_y=\triangle E \cos \theta.$$

As all the y—components are in the same direction, thus

$$E_y=\sum \triangle E \cos\theta=\sum \frac{\lambda \triangle x \cos\theta}{4\pi\varepsilon_0 r^2}=\int_a^{L+a} \frac{\lambda\, dx}{4\pi\varepsilon_0} \frac{\cos \theta}{r^2}.$$

Unfortunately, there are three variables (x, r and θ). To convert in one let us use the relation $r^2=x^2+b^2$ and $\cos \theta=b/r$.

$$\therefore\ E_y=\int_a^{L+a} \frac{\lambda b}{4\pi\varepsilon_0} \frac{dx}{(x^2+b^2)^{3/2}}=\frac{\lambda b}{4\pi\varepsilon_0}\left[\frac{1}{b^2}\frac{x}{(x^2+b^2)^{1/2}}\right]_a^{L+a}$$

$$=\frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{L+a}{[(L+a)^2+b^2]^{1/2}}-\frac{a}{(a^2+b^2)^{1/2}}\right]. \qquad \ldots(22)$$

The x—component of **E** can be obtained in the similar way.

$$E_x=\sum \triangle E \sin \theta=\int_a^{L+a} \frac{\lambda}{4\pi\varepsilon_0} \frac{x dx}{(x^2+b^2)^{3/2}}$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\left[-\frac{1}{[(L+a)^2+b^2]^{1/2}}+\frac{1}{(a^2+b^2)^{1/2}}\right] \qquad \ldots(23)$$

This relation shows that $E_x=0$ for $a=-L/2$ or on the points lying on the perpendicular bisector of the rod. For these points, the value of $\triangle E_x$ due to the left hand segment cancels the $\triangle E_x$ due to the right hand segment at the same distance.

Another interesting property of the result will be obtained when point P is very far from rod, *i.e.*, $b \gg L$. In this case

$$E_y=\frac{\lambda}{4\pi\varepsilon_0}\frac{L}{b^2}=\frac{q}{4\pi\varepsilon_0 b^2} \text{ and } E_x=0. \qquad \ldots(24)$$

This relation is same as due to a point charge $q(=\lambda L)$. It is expected, as the rod will appear to be a point charge when viewed from far away.

In the special case when the rod is of infinite length, the field can be obtained by integrating with in the limits $x=-\infty$ to $x=+\infty$ and the result will be

$$E_y=\lambda/2\pi\varepsilon_0 b \text{ and } E_x=0. \qquad \ldots(25)$$

Charged Loop. Consider a circular loop of radius a carrying a uniform charge λ per unit length. It is very difficult to compute **E** at any point in the space, however easy at points on its axis. Let us divide the loop in very small segments so that each segment can be assumed as a point charge. The electric field at a point P on the axis of the loop at a distance x from its centre due to the charge on the segment of length δl (Fig. 2.7) is given by

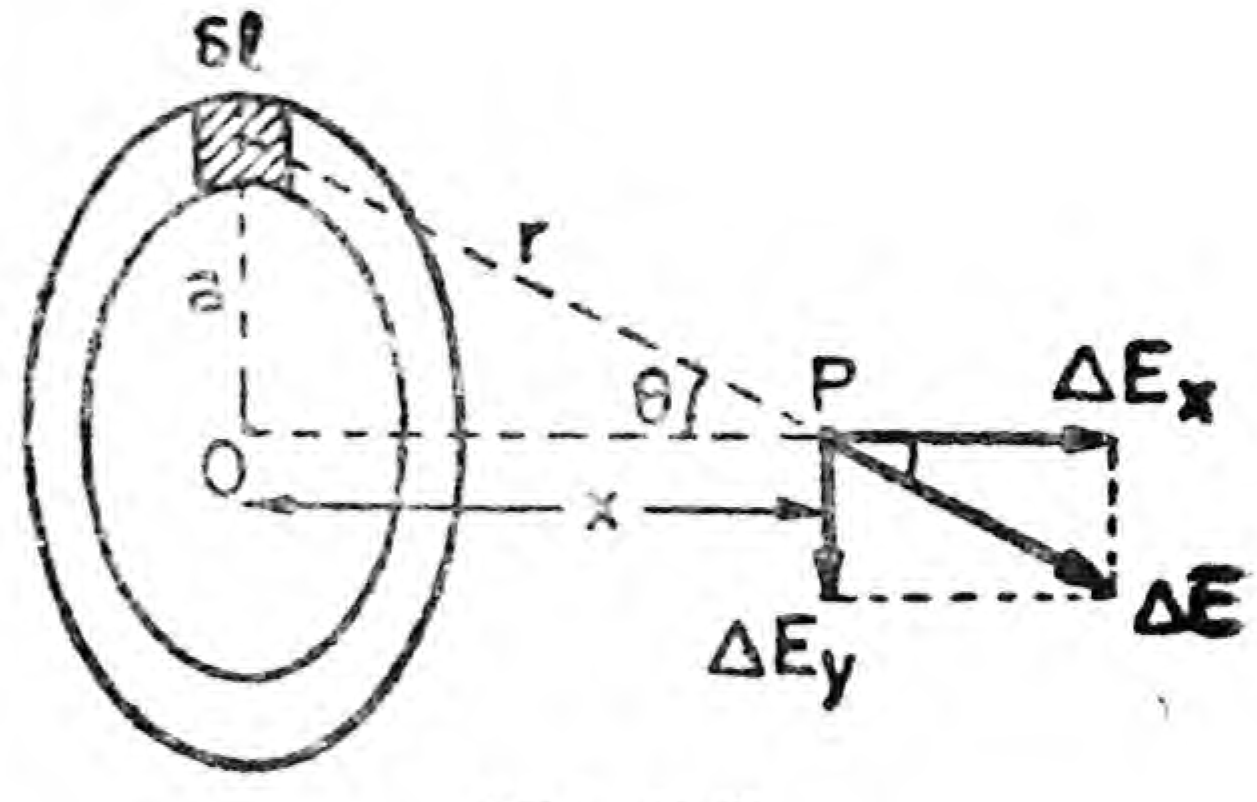


Fig. 2.7.

$$\Delta E=\frac{1}{4\pi\varepsilon_0}\frac{\lambda\delta l}{r^2}.$$

Its direction is along the line from δl to point P. The contributions to **E** of the various parts of the ring will be in different directions. The y-component ΔE_y cancels with the contribution due to the oppositely placed element of the loop and the net y-component is thus zero. The resultant field **E** will therefore be

$$E=\sum \Delta E_x=\int \frac{\lambda}{4\pi\varepsilon_0}\frac{dl}{r^2}\cos\theta.$$

Using relations $r^2=a^2+x^2$ and $\cos\theta=x/r$, we get

$$E=\frac{\lambda}{4\pi\varepsilon_0}\frac{x}{(a^2+x^2)^{3/2}}\int dl=\frac{\lambda}{4\pi\varepsilon_0}\frac{x.2\pi a}{(a^2+x^2)^{3/2}}$$

$$=\frac{qx}{4\pi\varepsilon_0(a^2+x^2)^{3/2}}. \qquad (\text{as } q=2\pi a\lambda) \qquad \ldots(26)$$

At $x=\infty$, the field will thus be same as due to a point charge.

(d) Infinite Plane Distribution of Charge. Let us find the field at a point P, a height z above an infinite plane distribution of charge of uniform charge density σ. For this, let us consider a charge element ΔS at any point on the plane at a distance r from point P with the co-ordinates $x, y, 0$. (Fig 2.8).

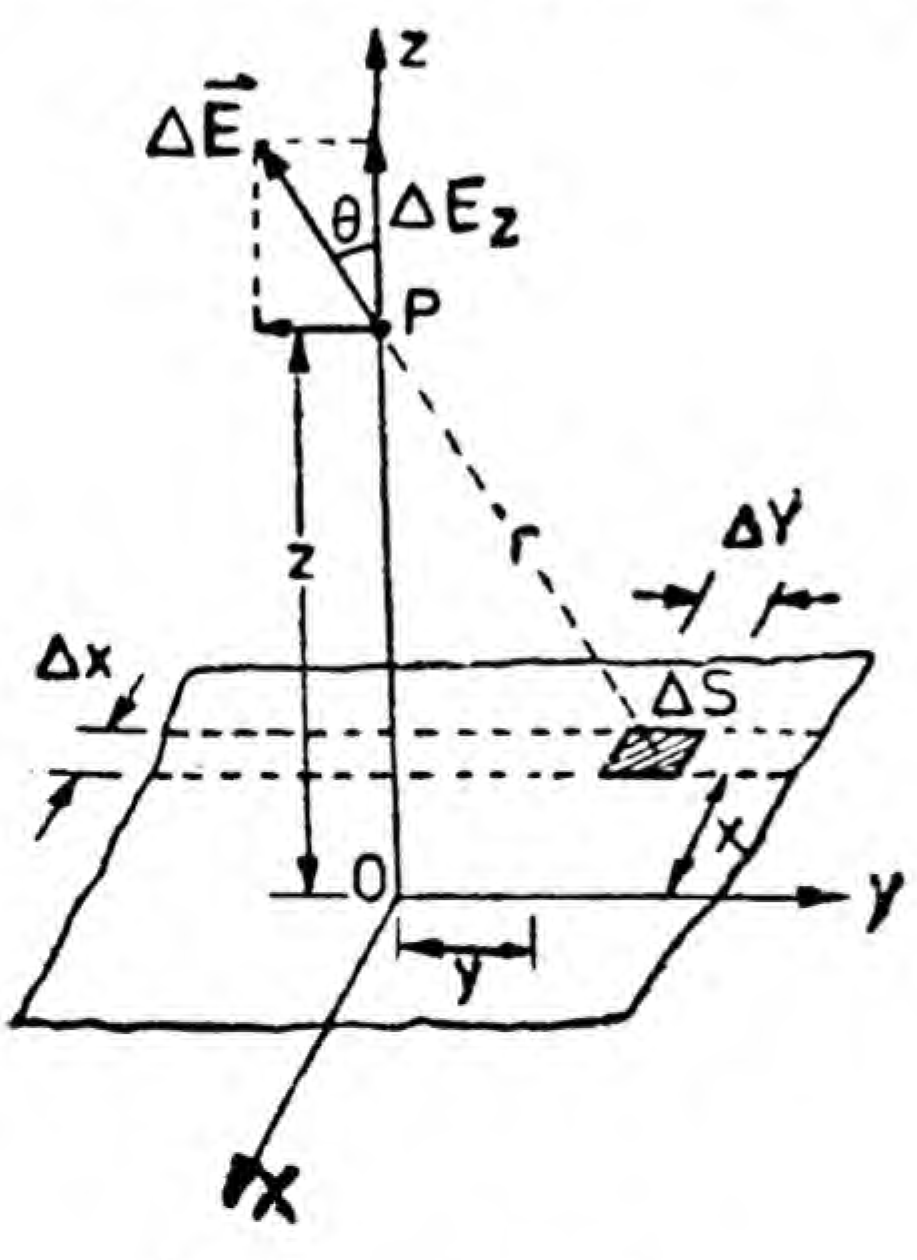


Fig. 2.8.

Hence field due to this element ΔS is given by

$$\Delta \mathbf{E}=\frac{1}{4\pi\varepsilon_0}\frac{\sigma\Delta S}{r^2}\hat{r}$$

$$\therefore\ \Delta E_z=\frac{\sigma}{4\pi\varepsilon_0}\frac{\Delta x\Delta y}{r^2}\times\frac{z}{r}$$

$$=\frac{\sigma z\Delta x\Delta y}{4\pi\varepsilon_0(x^2+y^2+z^2)^{3/2}}.$$

All the components along the z-direction are added while perpendicular components are cancelled due to the symmetry of the plane of infinite size. Hence the total electric field due to the infinite plane of charge

$$E=E_z=\frac{\sigma}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} dy\int_{-\infty}^{+\infty}\frac{z\,dx}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{\sigma}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} dy\,\frac{2z}{y^2+z^2}=\frac{\sigma}{2\varepsilon_0}. \qquad ...(27)$$

Thus we see that the field strength at a point outside an infinite plane distribution of charge is independent of the distance z of P from the plane and is in the opposite direction on the other side of the plane and hence suffers a discontinuity at the plane.

The above relation can also be obtained by dividing the plane in circular rings. We leave it as an exercise for the students.

(e) Charged Disc. Let us calculate the field strength on the axis of a disc of radius R, at a distance x from its centre O. For this let us divide the disc into annular rings. The field at a point P (Fig 2.9) due to charge element δs on the annular ring of radius a and of thickness δa is given by

$$\Delta\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\frac{\sigma\delta s}{r^2}\hat{r}.$$

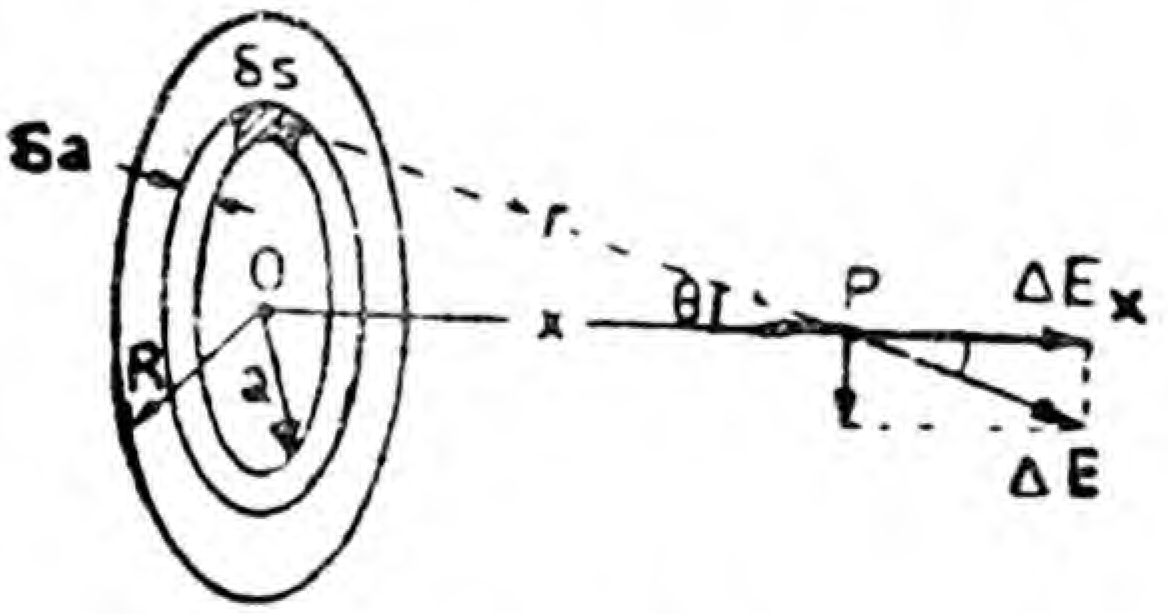


Fig. 2.9.

Its component along the axis of the disc or along the x direction

$$\Delta E_x=\frac{1}{4\pi\varepsilon_0}\frac{\sigma\delta s}{r^2}\frac{x}{r}.$$

For reasons of symmetry, the total field is along x-direction and is obtained by adding vectorially or by integrating. Assuming that the surface density of charge σ is a function of the distance from the centre of the disc, *i.e.*, $\sigma=ba$, where b is a constant of proportionality.

$$\therefore\ E=E_x=\frac{1}{4\pi\varepsilon_0}\int_0^R\frac{xba}{(a^2+x^2)^{3/2}}\int ds=\frac{bx}{2\varepsilon_0}\int_0^R\frac{a^2\,da}{(a^2+x^2)^{3/2}}$$

$$=\frac{bx}{2\varepsilon_0}\left[\log_e\frac{R+(R^2+x^2)^{1/2}}{x}-\frac{R}{(R^2+x^2)^{1/2}}\right] \qquad ...(28)$$

If the surface density of charge σ is constant, then

$$E=E_x=\frac{\sigma}{4\pi\varepsilon_0}\int_0^R\frac{x\,ds}{r^3}=\frac{\sigma}{4\pi\varepsilon_0}\int_0^R\frac{x\,2\pi a\,da}{(x^2+a^2)^{3/2}}$$

$$=\frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{(x^2+R^2)^{1/2}}\right]. \qquad ...(29)$$

2.5. EFFECT OF ELECTRIC FIELD ON A POINT CHARGE

The force **F** on a charged particle of mass m and charge q due to an electric field of strength **E** is given by

$$\mathbf{F}=q\mathbf{E}. \qquad \ldots(30)$$

This force will produce an acceleration in the charged particle, given by

$$\mathbf{a}=\mathbf{F}/m=(q/m)\mathbf{E}. \qquad \ldots(31)$$

This acceleration is of charged particle placed, not of that due to which the electric field **E** is. For example, the earth's gravitational field can not have any effect on the earth but only on a second body, say a stone, placed in that field.

2.6. LINES OF FORCE

When the vector **E** is known for all points in an electric field, the field is completely specified. The direction and magnitude of field at any point indicate how a small charge q_0 would begin to move if put there. By following this direction from point to point curves are obtained. These are termed as *lines of force*. Thus *the line of force in an electric field is a curve such that the tangent at any point on it gives the direction of the resultant electric field strength at that point. This is also the path on which a test charge will tend to move, if free to move.*

These lines are imaginary, their existence can be shown by sprinkling saw dust or gypsum salt in the electrostatic field. These particles acquire charges and place themselves along the lines of force. *The properties of electric lines of force are* :

1. No line of force originates or terminates in the space surrounding a charge. Every line of force is a continuous and smooth curve originating from a positive charge and ending on a negative charge.

2. The tangent to a line of force at any point gives the direction of **E** at that point.

3. They do not pass but leave or end on a charged conductor normally when the charges on the conductor are in equilibrium. Suppose the lines of force are not ⊥ to the conductor surface. In this situation the component of electric field ∥ to the surface would cause the electrons to move and would therefore give rise to a current. Since there is no current on the surface, hence lines of force are always ⊥ to the conductor surface.

4. Lines of force never intersect. If it happens then we say that at the point of intersection the electric intensity is zero, otherwise it would have to be tangential to two different curves at the same instance which is impossible.

5. The number per unit area crossing a surface at right angle to the field direction at every point is proportional to the electric intensity. Hence the lines of force are closely spaced where the intensity is large and are widely separated where the intensity is small.

6. They repel each other in the case of two like charges.

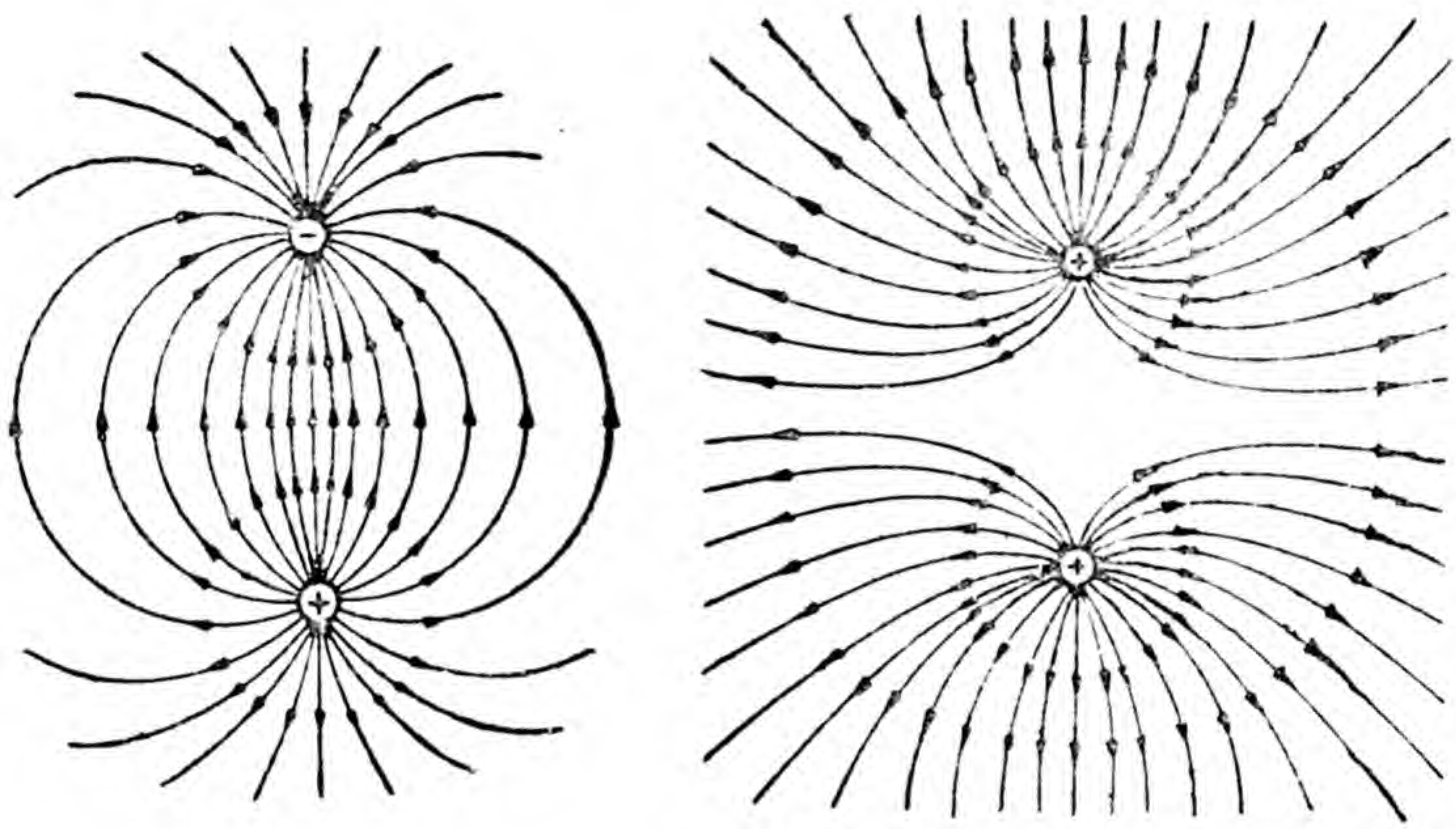


Fig. 2.10.

The direction of lines of force in two cases are shown in Fig. 2.10.

We define unit field (1 N/C) arbitrarily as corresponding to unit density of lines of force (one line/m^2). To determine the number of lines per unit charge, let us draw a spherical surface of radius r around the charge q as its centre. The electric field E at the surface is given by $E=q/4\pi\varepsilon_0 r^2$. As the density of lines of force is E, the total number of lines of force originating at q and crossing the spherical surface will be $4\pi r^2 E=q/\varepsilon_0$. Hence the *number of lines originating from a unit charge is $1/\varepsilon_0$*.

Electric lines of force are very useful as they give qualitative picture of the electric field distribution. There is a simple connection between the lines of force and inverse square law. We know that the density of lines at r meter from an isolated charge q is $q/4\pi\varepsilon_0 r^2$. As the area of the spherical surface subtended by a bundle of lines originating from the point charge is four times greater for a surface $2r$ meter away than for a surface at r meter distance. Thus the density of lines of force at $2r$ meter is just one quarter that at r meter in agreement with the value from inverse square law.

2.7. SOLID ANGLE

Solid angle is the analogue in three dimensions of the usual angle in two dimensions. In two dimensions, the unit angle is the angle subtended by an arc of length equal to its radius and is known as ***radian***. In three dimensions, a given area on a sphere subtends a certain solid angle at the centre. The unit solid angle is that subtended at the centre of a sphere by an area r^2 and is known as steradian. Since total surface area of the sphere is $4\pi r^2$, hence the total solid angle subtended at its centre is 4π steradians.

In Fig. 2.11, dS and dS' subtend the same solid angle. Thus the solid angle subtended by any surface dS at a point O a distance r away is given by

$$d\omega = dS \cos\theta / r^2, \qquad \ldots(32)$$

where $dS \cos\theta$ $(=dS')$ is the projection of the surface dS perpendicular to the radius vector r from the point O.

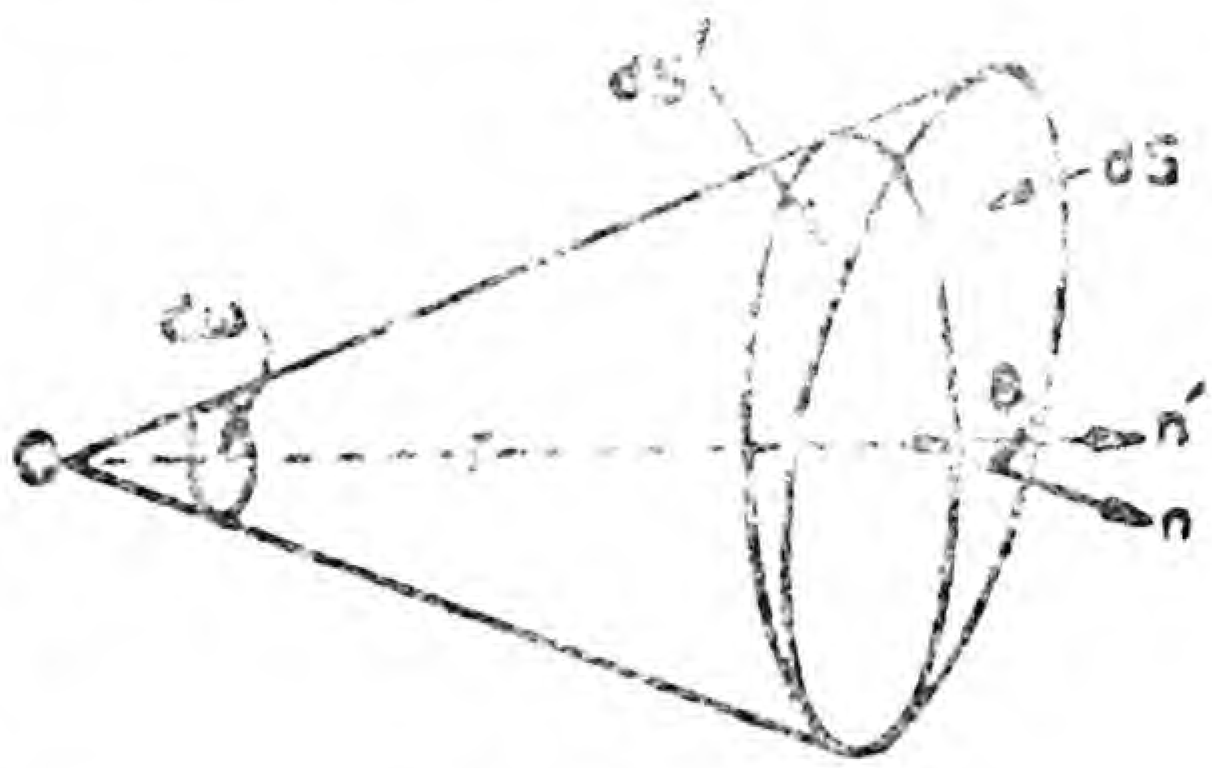


Fig. 2.11.

2.8. GAUSS'S LAW

Consider a single positive point charge q surrounded by a closed surface of arbitrary shape, as shown in Fig. 2.12. The electric

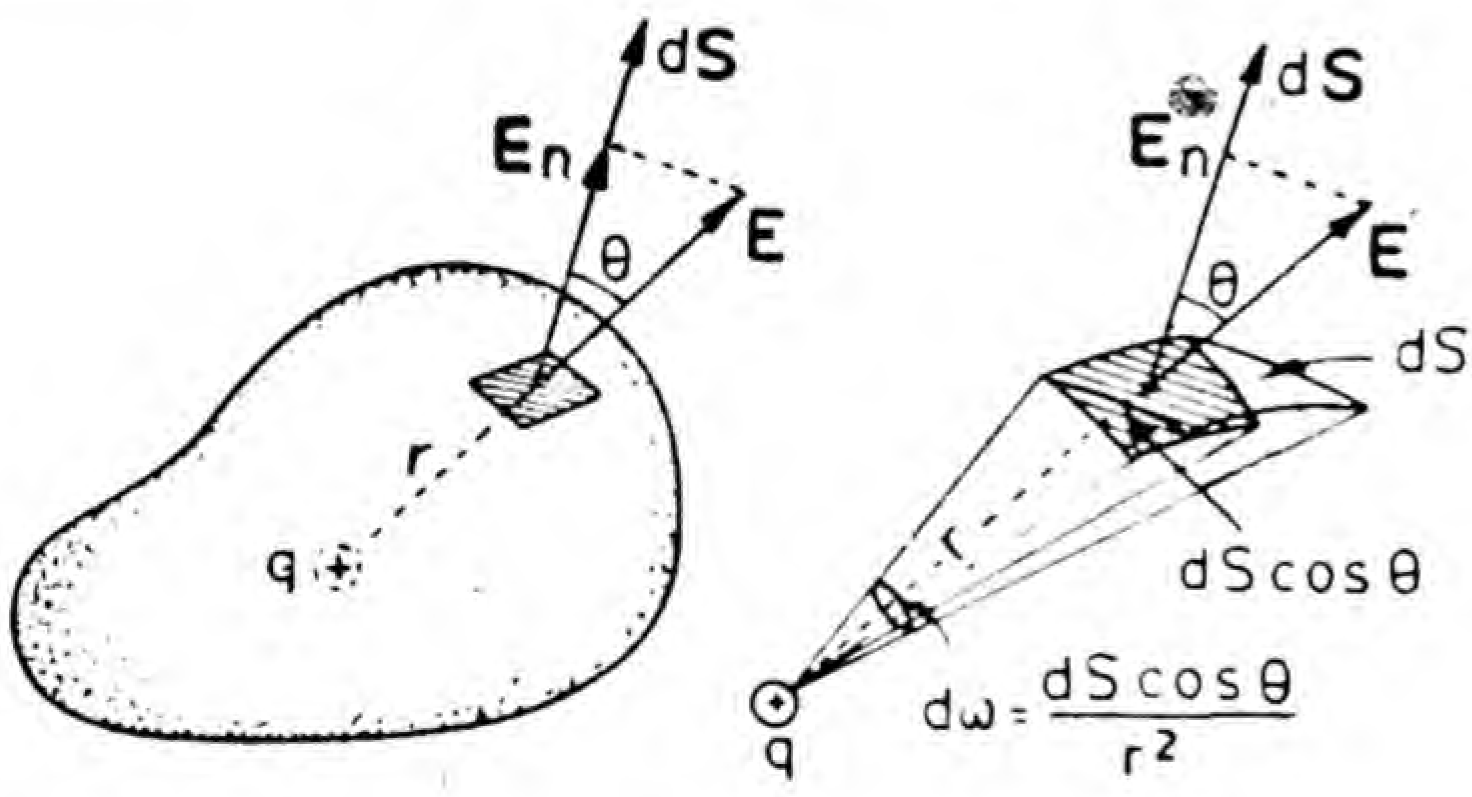


Fig. 2.12.

intensity **E** at every point of the surface is directed radially outward from the charge. Let us consider any sufficiently small area dS of the surface for which **E** can be considered to have same magnitude and direction. If θ be the angle between **E** and the outward normal to the surface at any point of this area. The product E_n, the component of **E** normal to the surface, and the area dS is

$$E_n\, dS = E \cos\theta\, dS = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \cos\theta\, dS.$$

As $dS \cos\theta / r^2$ can be replaced by the term solid angle $d\omega$ subtended at the charge q by the area dS, hence

$$E_n\, dS = (q/4\pi\epsilon_0)\, d\omega.$$

If the surface is continuous we can integrate both sides of this equation over the entire closed surface and so we have

$$\oint_S E_n\, dS = \frac{q}{4\pi\varepsilon_0} \oint d\omega.$$

Regardless of the shape or size of the closed surface, $\oint d\omega$=Total solid angle surrounding the charge $q=4\pi$,

$$\therefore \qquad \oint_S E_n\, dS = \frac{q}{4\pi\varepsilon_0}\, 4\pi = \frac{q}{\varepsilon_0}.$$

As the product $E_n\, dS = E \cos\theta\, dS = \mathbf{E}.d\mathbf{S}$, hence we have

$$\oint_s \mathbf{E}.d\mathbf{S} = q/\varepsilon_0. \qquad \ldots(33)$$

If the point charge is negative the field **E** is directed radially inward, the angle θ is greater than 90° or E_n is negative. The surface integral is thus negative. Hence the form of Eq.(33) is correct whatever be the sign of the charge.

When the point lies outside the surface, every elementary cone from O cuts the surface an even number of times. Let us draw a cone of solid angle $d\omega$ at point O, intercepting areas δS_1, δS_2, δS_3, δS_4, δS_5 and δS_6 at points A, B, C, D, E and F respectively. The corresponding outward normal over these areas are shown in Fig. 2.13. The angles θ_1, θ_3 and θ_5 are obtuse and θ_2, θ_4 and θ_6 are acute. Thus

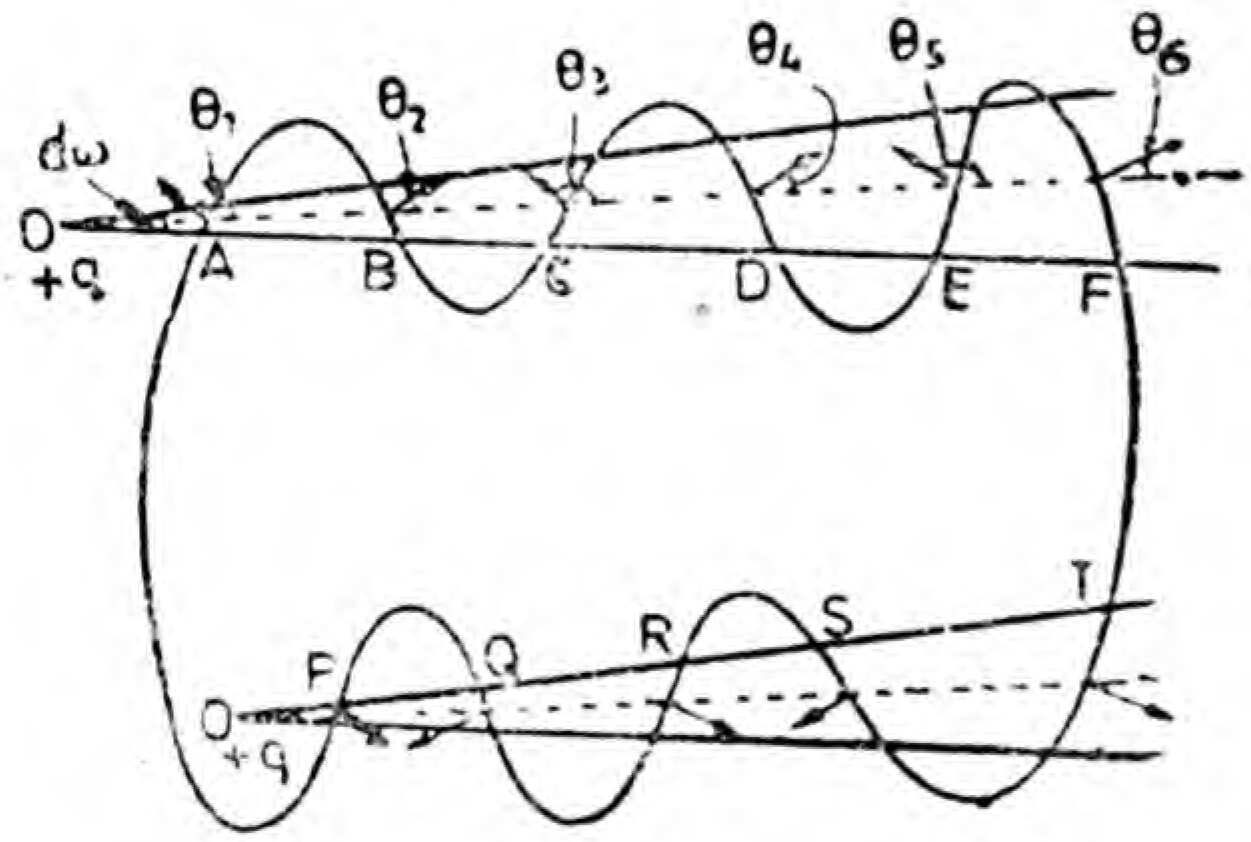


Fig. 2.13.

$$d\omega = \frac{\delta S_1 \cos(\pi - \theta_1)}{r_1^2} = -\frac{\delta S_1}{r_1^2} \cos\theta_1 = \frac{\delta S_2}{r_2^2} \cos\theta_2 = \text{etc.}$$

$$\therefore \quad \delta S_1 \cos\theta_1 = -r_1^2\, d\omega\, ;\ \delta S_2 \cos\theta_2 = r_2^2\, d\omega\, ;\ \ldots\text{etc.}$$

Hence the contribution from these six (say) areas to the surface integral is

$$E_1 \cos\theta_1\, \delta S_1 + E_2 \cos\theta_2\, \delta S_2 + E_3 \cos\theta_3\, \delta S_3 + E_4 \cos\theta_4\, \delta S_4 + \ldots$$

$$= \frac{q}{4\pi\varepsilon_0 r_1^2}(-r_1^2 d\omega) + \frac{q}{4\pi\varepsilon_0 r_2^2}(r_2^2\, \delta\omega) + \frac{q}{4\pi\varepsilon_0 r_3^2}(-r_3^2\, d\omega) + \ldots$$

$$= \frac{q}{4\pi\varepsilon_0}[-d\omega + d\omega - d\omega + d\omega - d\omega + d\omega] = 0. \qquad \ldots(34)$$

But the charge inside the closed surface is zero, hence the Eq. (33) holds good whether the charge inside the surface is +ve, —ve or zero.

Any arbitrary distribution of charges can be assumed as a system of point charges. We can use Eq. (33) for each point charge and sum over all charges. In this case charge q is equal to Σq or more accurately $\int \rho \, dV$. Thus

$$\oint_S \mathbf{E} . d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho dV. \quad ...(35)$$

This is the equation which expresses Gauss's law : *The surface integral of the normal component of electric field* **E** *over any closed surface in an electrostatic field equals* $1/\varepsilon_0$ *times the total charge enclosed by the surface.*

The left hand side of Eq. (35) is called the flux of **E** across the surface and is represented by Φ. Thus

$$\Phi = \oint_S \mathbf{E} . d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho dV. \quad ...(36)$$

The term flux is borrowed from hydrodynamics, where a similar integral in which **E** is replaced by the velocity vector **v** gives the net flow of fluid across a surface. The Gauss's law can now be stated as : *the outward flux of electric field across a closed surface equals* $1/\varepsilon_0$ *times the net charge contained in the volume enclosed by the surface.*

In terms of electric displacement density **D** (to be defined later), which is equal to $\varepsilon_0 \mathbf{E}$, the Gauss's law can be stated as : *the net outward electric displacement through a closed surface is equal to the net charge contained in the volume enclosed by the surface, i.e.,*

$$\oint_S \mathbf{D} . d\mathbf{S} = \int_V \rho \, dV. \quad ...(37)$$

Gauss's law can also be written in the other form by using divergence theorem $\oint_S \mathbf{D} . d\mathbf{S} = \int_V \nabla . \mathbf{D} \, dV$,

as

$$\int_V \nabla . \mathbf{D} \, dV = \int_V \rho dV. \quad ...(38)$$

This holds for any volume whatsoever. As the volume considered is reduced to an elementary volume, this becomes the point relation

$$\nabla . \mathbf{D} = \rho, \quad ...(39)$$

i.e., at every point in a medium the divergence of electric displacement is equal to the charge density at that point.

Graphical Interpretation of Gauss's Law. For a flow field the flux is measured by the number of stream lines that cut through such a surface. As the number of electric lines of force per unit area at right angles to their direction is proportional to E, the surface integral of **E** over a closed surface is proportional to the total number of lines of force crossing the surface in an outward direction. The total number of lines of force is also proportional to the net charge within the surface.

For closed surfaces Φ is positive if the lines of force point outward everywhere (**E** is outward every where, $\theta < 90°$ and $\mathbf{E}.d\mathbf{S}$ will be positive) and negative if they point inward (**E** is inward every where, $\theta > 90°$ and $\mathbf{E} \,.\, d\mathbf{S}$ will be negative). Let us consider two equal and opposite point charges and their lines of forces as shown in Fig. 2.14. From the statement just given, Φ is positive for surface S_1 and is negative for surface S_2. Surface S_4 encloses both charges. The net number of lines of force crossing in an outward direction is zero and the net charge inside the closed surface is also zero. Similarly in surface S_3, number of lines of force in inward direction is same as those in outward direction or the net number of outward lines of force is zero. The charge inside this surface is also zero. In this way it is clear that Gauss's law is correct whatsoever be the charge inside the closed surface.

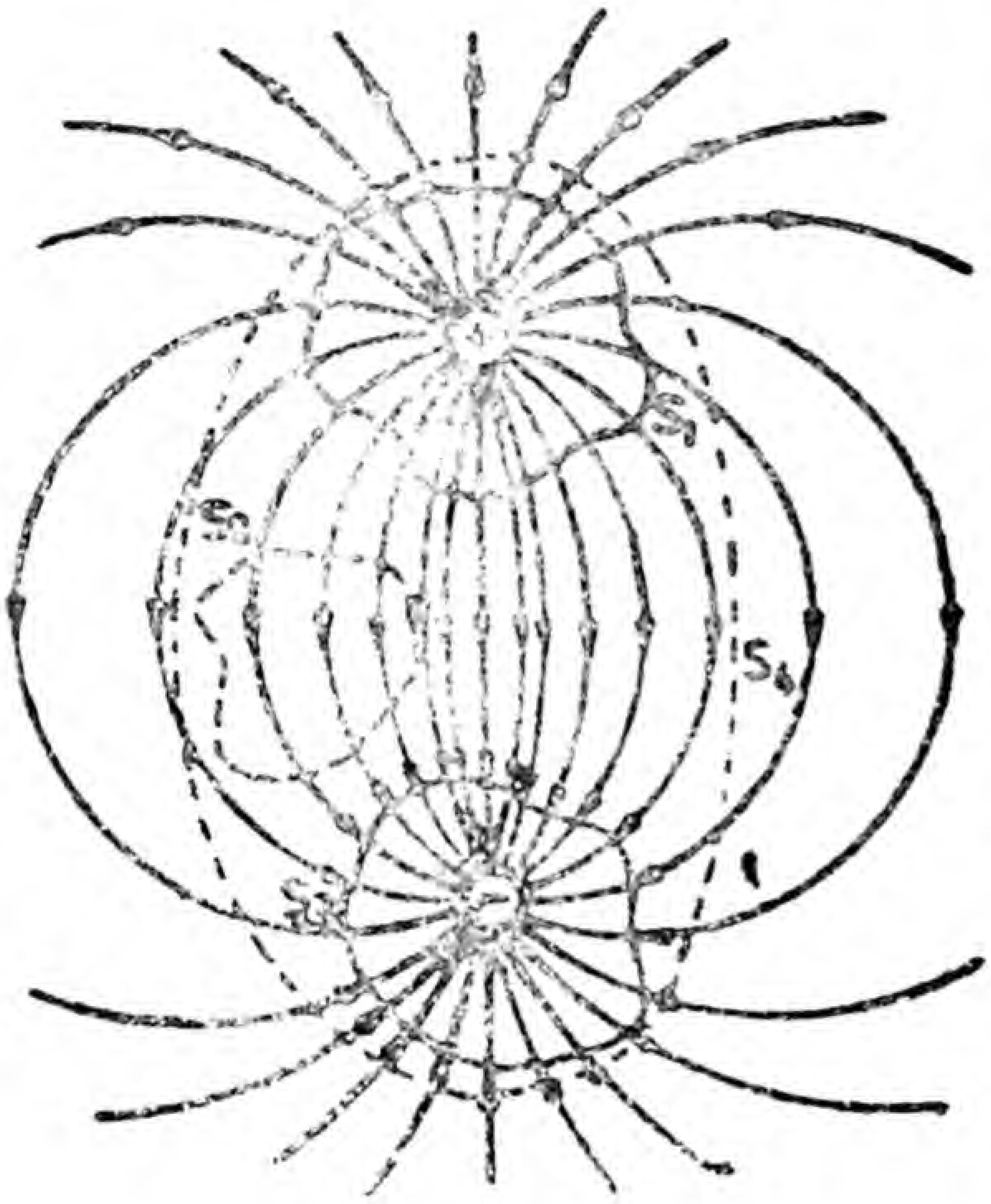


Fig. 2.14.

2.9. APPLICATIONS OF GAUSS'S LAW

As Gauss's law does not provide expression for **E** but provides only for its surface integral. To calculate **E** we must choose surface in such a way that the surface integral may be replaced by a product of which **E** is a factor. Let us discuss few simple cases :

(A) Point charge—The electric field due to a point charge is everywhere radial. Its magnitude is the same at all points at the same distance *r* from the charge. Hence we select gaussian surface (*a closed surface drawn in an electric field*) a spherical surface. It is also clear from Fig. 2.15 that both **E** and $d\mathbf{S}$ (direction of which is always perpendicular to the surface itself) at any point on the gaussian surface are directed radially outward. Hence

$\oint_S \mathbf{E} \cdot d\mathbf{S} = \oint_S E_n dS = ES = 4\pi r^2 E.$

But from Gauss's law it should be $1/\varepsilon_0$ times the total charge inside the gaussian surface. Hence

$$4\pi r^2 E = q/\varepsilon_0$$

or $$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \quad ...(40)$$

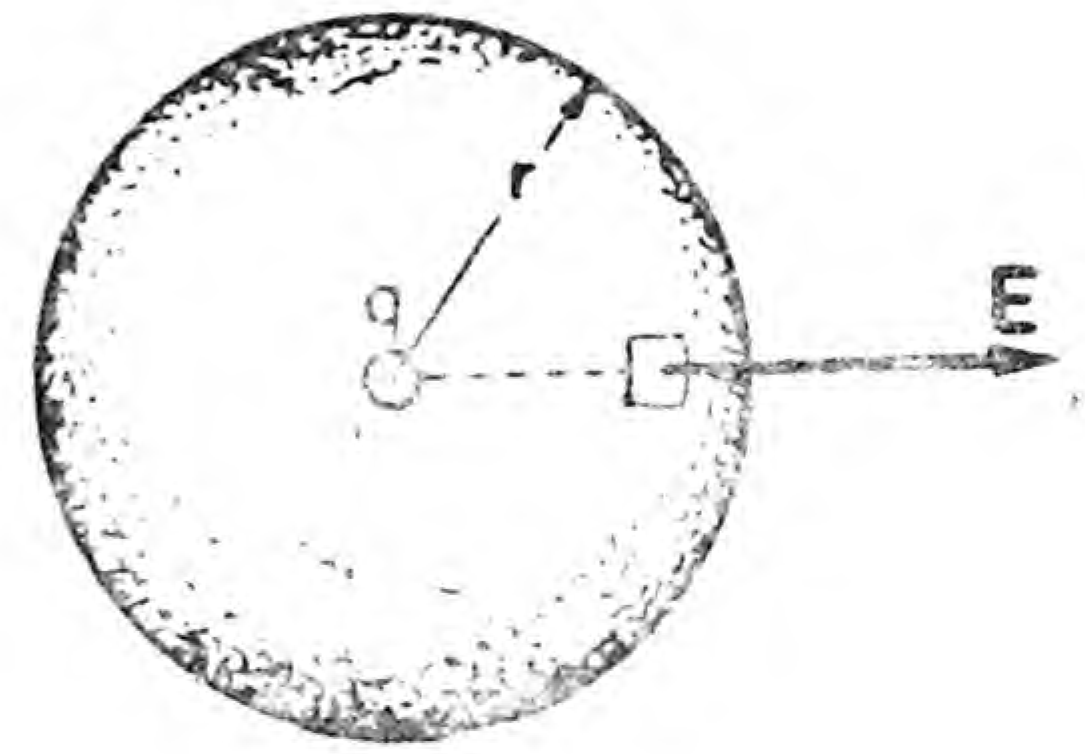


Fig. 2.15.

The force on the point charge q_0 if placed on this surface, *i.e.*, at a distance r from point charge q is then

$$F = q_0 E = qq_0/4\pi\varepsilon_0 r^2.$$

The force is acting in radially outward direction, hence can be represented vectorially as $\mathbf{F} = (qq_0/4\pi\varepsilon_0 r^2)\, \hat{r}$ ...(41)

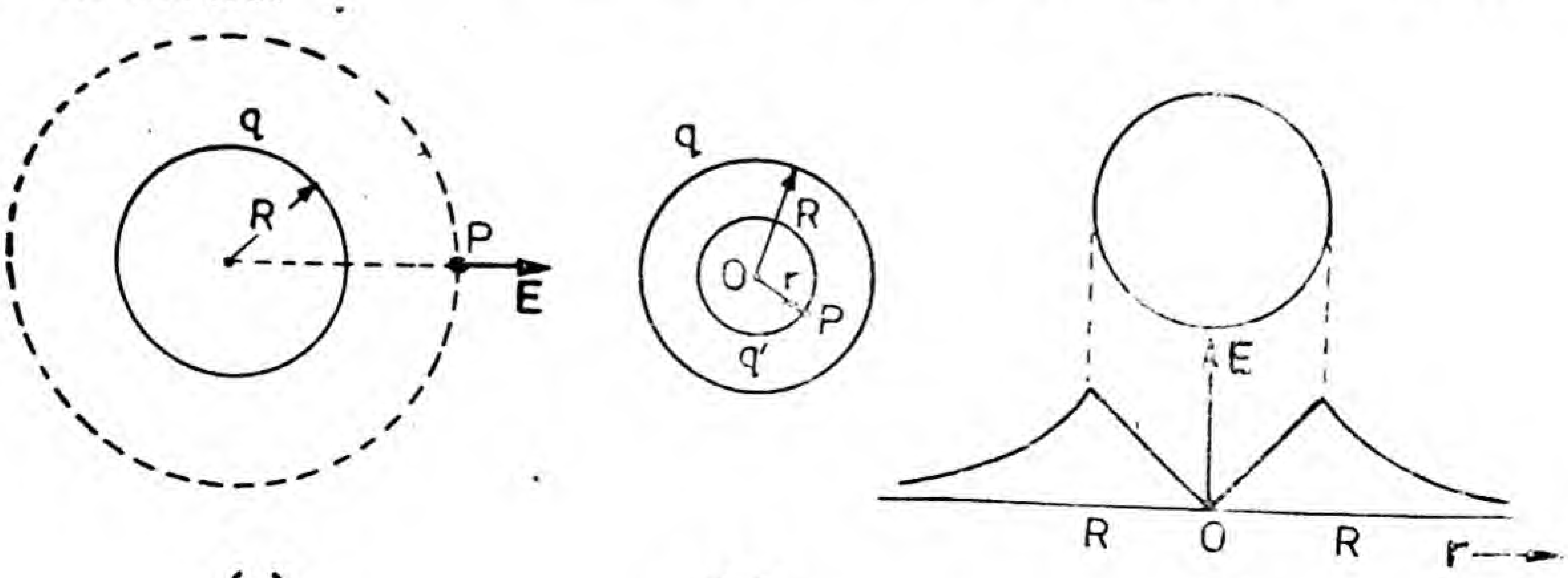
It is nothing but *Coulomb's law.*

(B) Uniformly charged sphere—In the uniformly charged sphere of radius R and charged to q units, the charge density ρ at every point is constant. It is evident that the field due to a spherical charged surface at an external point P has the same symmetry as that of a point charge. Thus we can construct gaussian surface by drawing a concentric sphere of radius OP with centre O. At all points of this sphere the magnitude of electric field is the same and its direction is perpendicular to the surface. Thus

$\oint_S \mathbf{E}.d\mathbf{S} = \oint_S E_n dS = E 4\pi r^2$ and from Gauss's law

$$4\pi r^2 E = \frac{q}{\varepsilon_0} \text{ or } E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}. \quad ...(42)$$

Hence the *electric field at any external point due to a uniformly charged sphere is the same as if the total charge is concentrated at its centre.*



(a) (b) (c)

Fig. 2.16.

At the surface of sphere $r=R$, the electric field

$$E=\frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}. \qquad ...(43)$$

For the field inside the charged sphere let us draw gaussian surface of radius $r(<R)$, as shown in Fig. 2.16 (b). Thus

$$\oint_S \mathbf{E}.d\mathbf{S}=\oint_S E_n dS=4\pi r^2 E$$

and from Gauss's law

$$4\pi r^2 E=q'/\varepsilon_0,$$

where q' is that part of charge which is with in the sphere of radius r. The part of q that lies outside this sphere makes no contribution to E at the surface of sphere of radius r. If ρ is the volume density of the charge which is constant, then

$$q'=\frac{4}{3}\pi r^3\rho=\frac{4}{3}\pi r^3\left(\frac{q}{\frac{4}{3}\pi R^3}\right)=\left(\frac{r^3}{R}\right)^3 q.$$

$$\therefore 4\pi r^2 E=\frac{q}{\varepsilon_0}\left(\frac{r}{R}\right)^3 \quad \text{or } E=\frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}.$$

In vector from

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\mathbf{r}. \qquad ...(44)$$

From Eqs (42), (43), and (44), it is clear that the electric field due to a uniformly charged sphere at the surface is maximum and is zero at the centre [Fig. 2.16 (c)].

(C) An Isolated uniformly charged spherical conductor. In an isolated charged spherical conductor any excess charge on it is distributed uniformly over its outer surface and there is no charge inside it. Hence this problem is same as that of charged spherical shell or hollow sphere. As in the previous case, the field at external points has the same symmetry as that of a point charge so we can construct a gaussian surface of radius $r>R$. The electric intensity calculated, as in the previous case, is given by the relation

$$E=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}, \qquad ...(45)$$

It is same as due to the uniformly charged sphere or the point charge of same magnitude if placed at the centre.

At the surface $r=R$,

$$E=\frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}. \qquad ...(46)$$

At all points inside the charged spherical conductor or hollow spherical shell, electric field **E**=0 as there is no charge inside such a sphere.

(D) Charged cylindrical conductor of infinite length. If the cylinder (say a wire) is long and of radus R and we are not too near to either end, then by symmetry the lines of force outside the cylinder are radial and perpendicular to its surface. The electric field has the same magnitude at all points at the same radial distance. Let us draw an imaginary co-axial cylinder (a gaussian surface) of length l through any point at a distance r from the axis of the cylinder as shown in Fig 2.17. Lines of force are parallel to the circular caps, hence there is no component of electric field normal to the end faces (the circular caps). Thus flux through these faces is zero and the total flux is due to curved surface only. If λ is the charge per unit length, *i.e.*, linear charge density, the charge with in the gaussian surface will be λl. Thus for the curved surface

$$\oint_S \mathbf{E}.d\mathbf{S}=\oint_S EdS = E2\pi rl$$

From Gauss's law

$$E\, 2\pi rl=q/\varepsilon_0=\lambda l/\varepsilon_0$$

or $$E=\lambda/2\pi\varepsilon_0 r. \qquad ...(47)$$

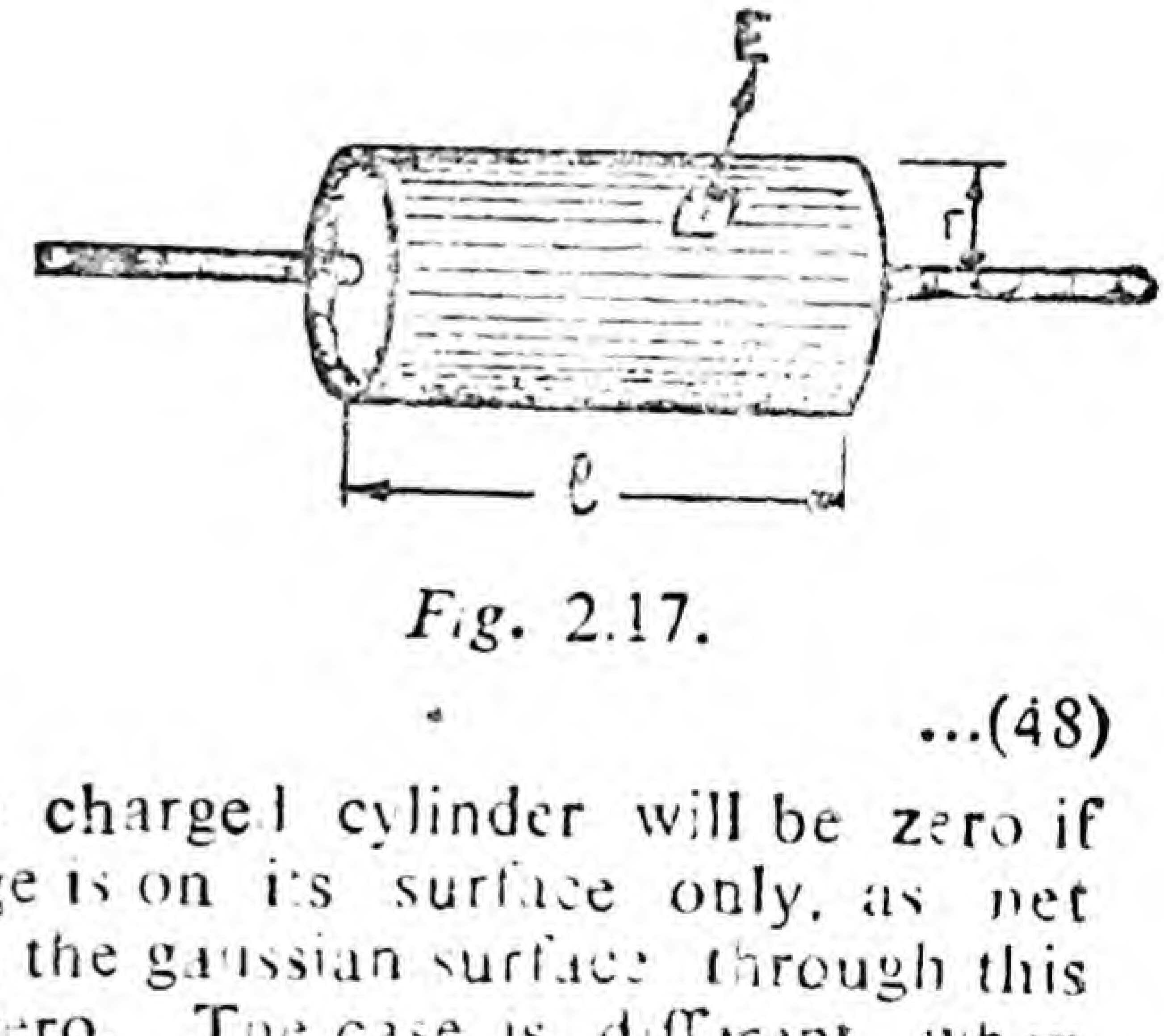


Fig. 2.17.

If the charge is distributed throughout the volume and charge density is ρ, then

$$2\pi rl\, E=\rho(\pi R^2 l)/\varepsilon_0$$

or $$E=\rho R^2/2\varepsilon_0 r. \qquad ...(48)$$

The electric field inside the charged cylinder will be zero if the charge is on its surface only, as net charge in the gaussian surface through this point is zero. The case is different when the electric charge is distributed uniformly with in the cylinder of radius R. To find **E** at an inner point P, a distance r apart from the axis of the cylinder, let us draw a gaussian surface passing through P, which is a cylinder of length l and of radius r (Fig 2.18). As the flat surfaces do not contribute to the flux and the flux is due to the curved surface for which **E** is $\perp$ to the surface or $\parallel$ to the surface vector $d\mathbf{S}$. Hence from Gauss's law we have

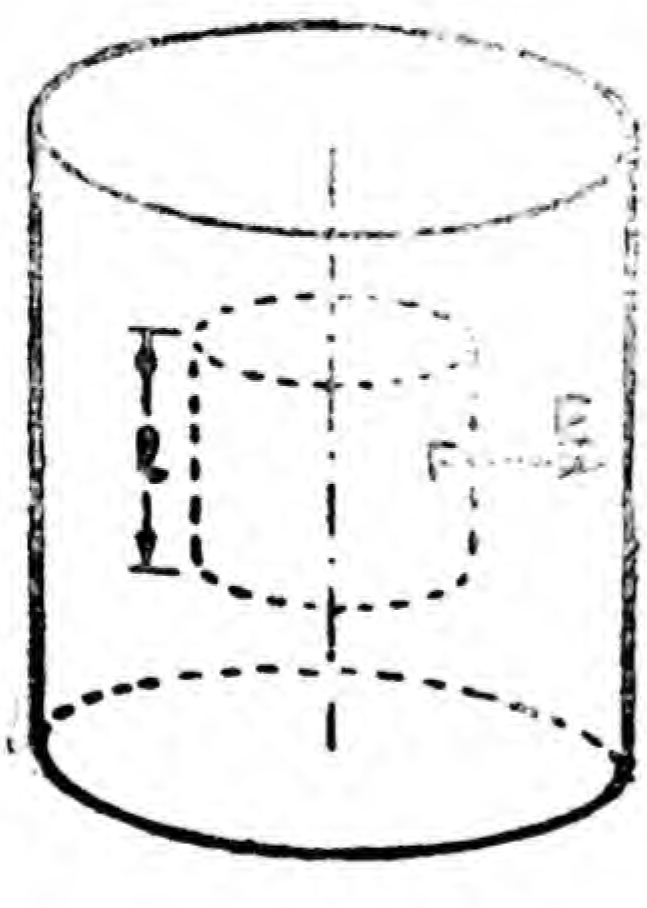


Fig. 2.18.

$$\oint_S \mathbf{E}.d\mathbf{S}=\oint_S EdS=E\, 2\pi rl=q'/\varepsilon_0.$$

The charge q' inside this gaussian surface$=\pi r^2 l\rho$

$$\therefore\ E2\pi rl=\pi r^2 l\rho/\varepsilon_0 \text{ or } E=r\rho/2\varepsilon_0. \qquad ...(49)$$

Thus the electric intensity at a point inside an infinite uniformly charged cylinder is radially directed and varies as the distance from its axis. At $r=R$, relations (48) and (49) give the same result. The field decreases at points outside.

Here we see that the electric field due to a charged cylinder does not depend upon its radius, hence we can say that it is same as though the charge on the cylinder were concentrated in a line along its axis. It is right to state that the electric field so calculated is due to the entire charge on the cylinder although only a portion of total charge is used when we apply Gauss's law. We can very well understand that the existence of the entire charge on the cylinder is taken into account indirectly by considering the symmetry of the problem. If the cylinder is of small length we can not conclude that the field at one end of cylinder is same as at the centre or the field is symmetric. It means that the lines of force are everywhere not perpendicular to the cylinder. In this way the entire charge is used indirectly by taking field symmetry.

(E) Infinite plane sheet of charge. Fig. 2.19 shows a portion of a flat thin sheet, infinite in size with the constant surface charge density σ. By symmetry, since the sheet is infinite, the field must have the same magnitude and the opposite directions at two points equidistant from the sheet on opposite sides. To solve the given problem let us draw a cylinder A (gaussian surface) with one end on one side and other end on the other side and of cross sectional area S. No lines of force cross the cylindrical wall, *i.e.*, the component of **E** normal to curved surface is zero. Thus the surface integral of **E**. d**S** over the entire surface of cylinder is reduced to the surface integral due to side surfaces. Hence Eq. (33) becomes

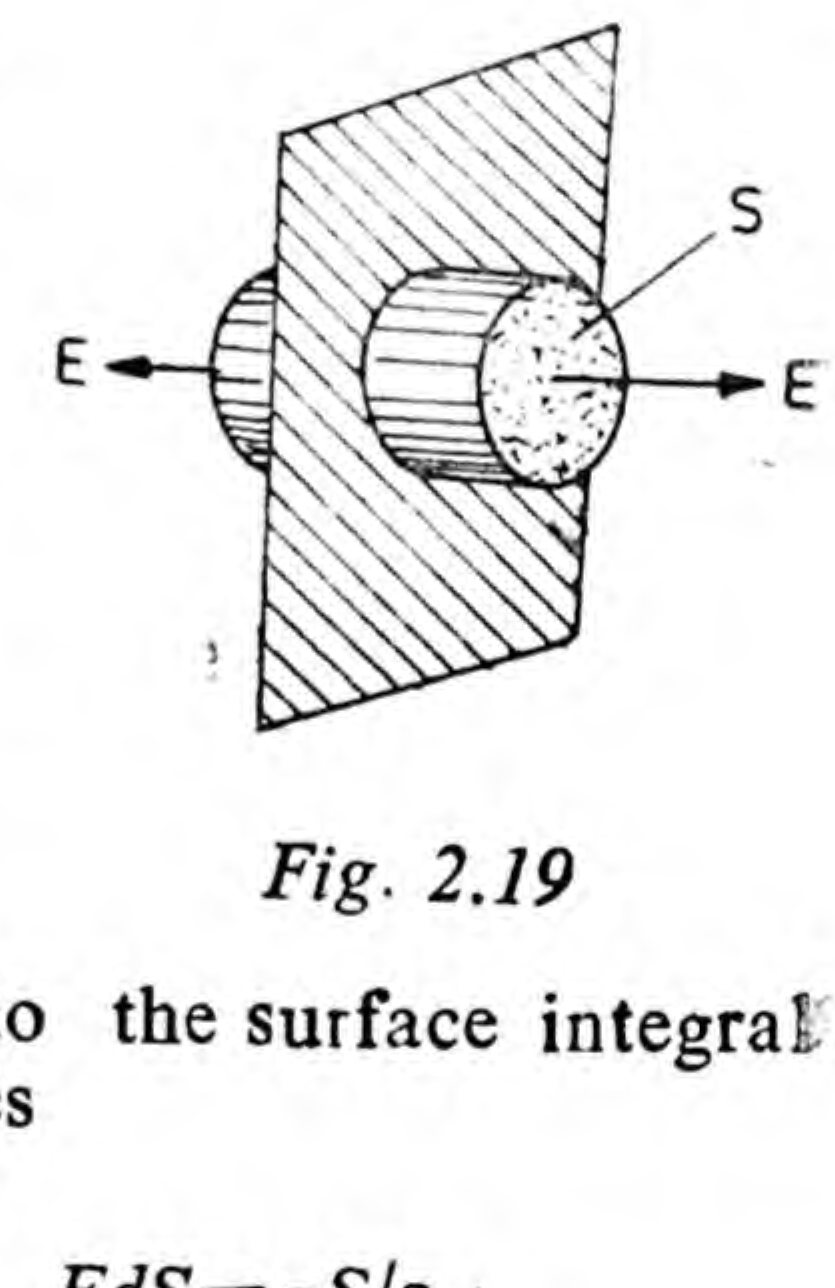


Fig. 2.19

$$\oint_S \mathbf{E}.d\mathbf{S}=q/\varepsilon_0 \text{ or } \oint_{S_1} EdS+\oint_{S_2} EdS=\sigma S/\varepsilon_0.$$

$$\therefore \quad 2ES=\sigma S/\varepsilon_0 \text{ or } E=\sigma/2\varepsilon_0. \quad \ldots(50)$$

Thus we see that the magnitude of the field is independent of the distance from the sheet. Practically an infinite sheet of charge does not exist. *These results are correct for real charge sheets if points under consideration are not near the edges and the distances from the sheet are small compared to the dimensions of sheet.*

(F) Infinite charged conducting plate. When a charge is given to a conducting plate, it distributes itself over the entire outer surface of the plate. The surface density of charge σ is uniform and is the same on both surfaces if plate is of uniform thicknses and of infinite size. The electric field at any point thus arises from the superposition of the fields of two sheets of charge, one on each surface of the plate. By symmetry it is very easy to say that the field is perpendicular to the plate and is in outward direction if the charge on the plate is positive and uniform. Draw a cylinder through any point P, as shown in Fig. 2.20, normal to the plate surface and cross sectional area S. The other end face of the cylinder lies inside the conductor. This face does not contribute to the flux as the field inside the conductor is zero. Curved surface of the cylinder also does not contribute to the flux as **E** is parallel to the curved surface. Thus the only face is the outside end face which contributes to the flux as **E** is ⊥ to this surface. Hence from Gauss's law

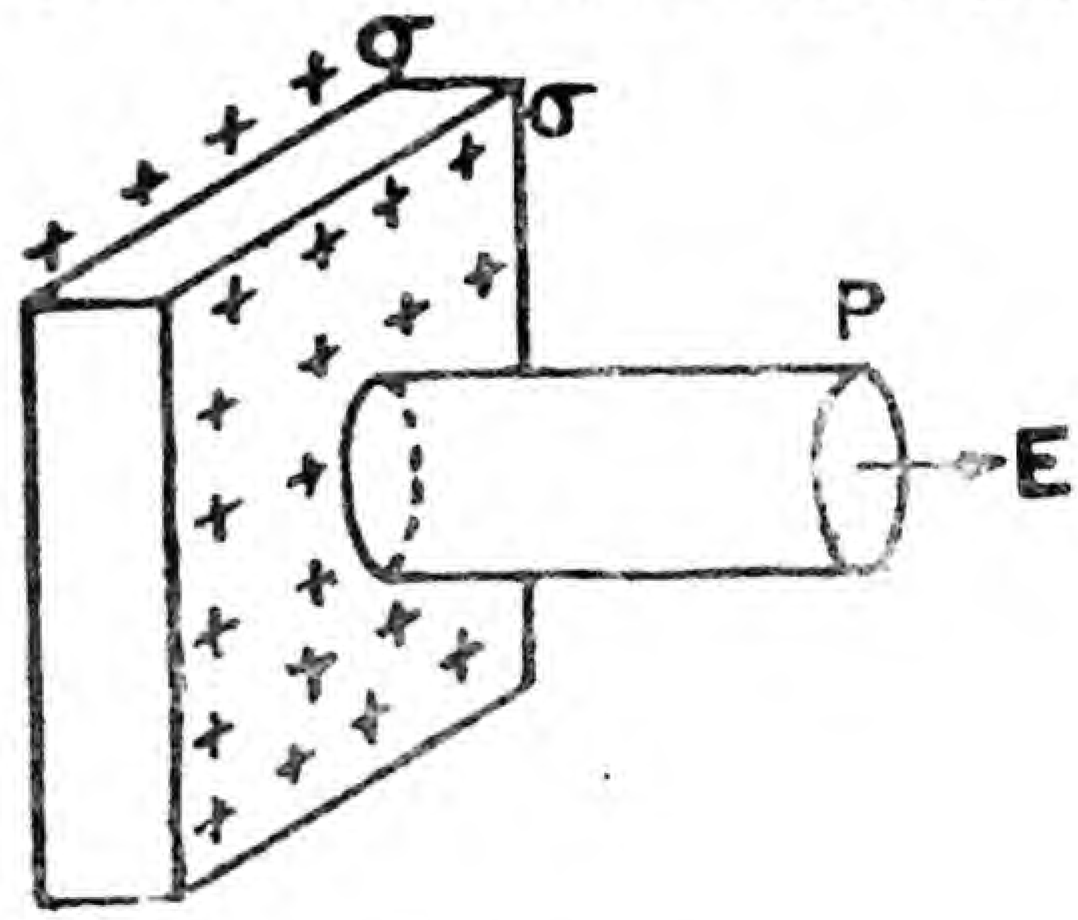


Fig. 2.20.

$$\oint_s \mathbf{E}.\, d\mathbf{S}=q/\varepsilon_0 \text{ or} \oint_s EdS=ES=\rho S/\varepsilon_0$$

$$\therefore \qquad E=\sigma/\varepsilon_0. \qquad \ldots(51)$$

This result shows that the field due to charged conducting plate is twice the field due to plane sheet of charge. It also has same limitations.

(G) Electric intensity between oppositely charged parallel plates. When two plane parallel infinite conducting plates, separated by a distance, are given equal and opposite charges, the field is uniform in the space between them. There is a small quantity of charge on the outer surfaces of the plates and a certain spreading or fringing (edge effect) of the field at the edges of the plates. We can assume fringing negligible as the plates are of the infinite size. Thus the field between plates can be taken as uniform.

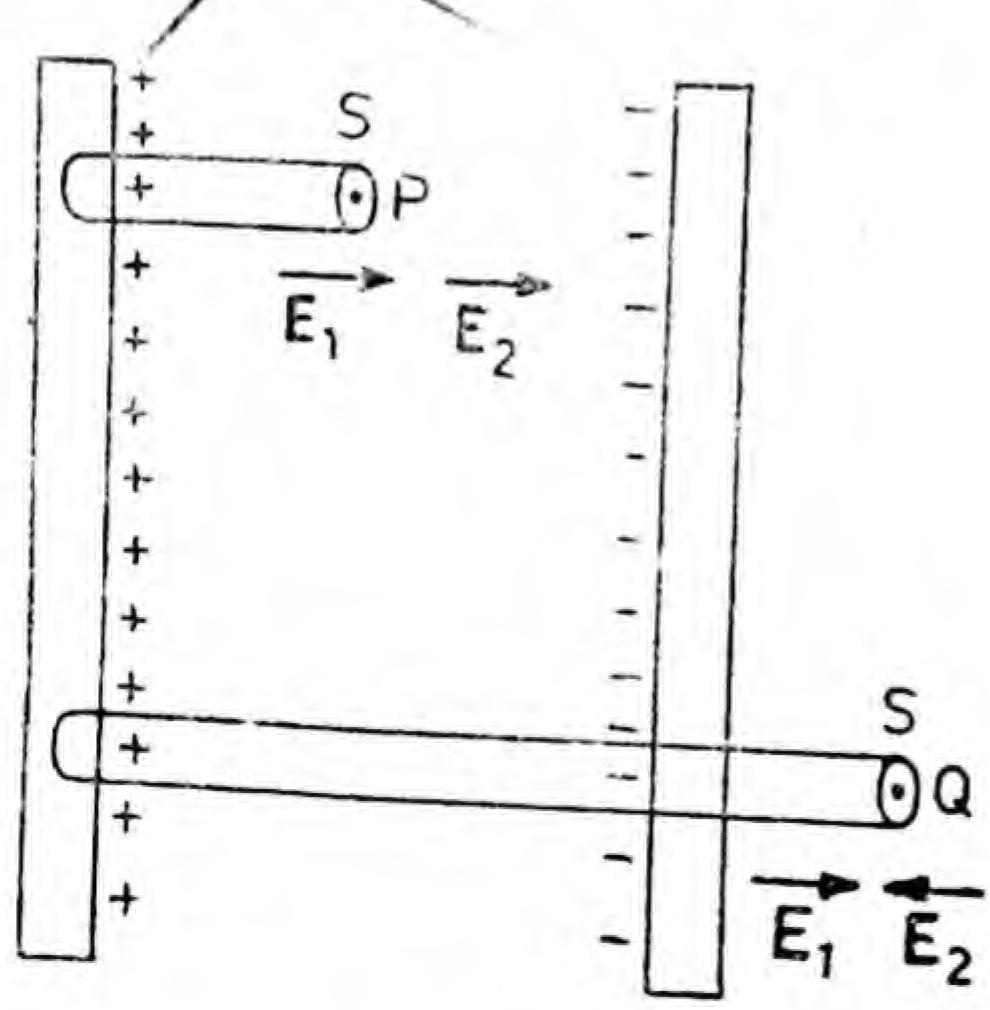


Fig. 2.21.

Draw a normal cylinder through a point P, between two plates and let the other end of this cylinder lying in the left conducting plate and crossing an area S. The problem is same as with an infinite charged conducting plate and the electric field at point P is thus given by

$$E=\sigma/\epsilon_0. \quad \text{...(52)}$$

To find the electric intensity at an outside point Q, draw the cylinder through the point Q whose second end face lying on the first conductor. It is also clear that the surface of the cylinder contributing to the flux is the end face passing through the point Q. Hence from Gauss's law

$$\oint_S \mathbf{E}.\, d\mathbf{S}=\oint_S EdS=ES=(\sigma S-\sigma S)/\varepsilon_0$$

or
$$E=0. \quad \text{...(53)}$$

The electric field at any point due to this system of plates may be written as the superposition of the fields due to infinite plane of sheets of charge. Therefore at point P, the electric field

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2 \text{ or } E=E_1+E_2$$

$$\therefore \quad E=\sigma/2\epsilon_0+\sigma/2\varepsilon_0=\sigma/\varepsilon_0.$$

For point Q, $\mathbf{E}_1$ and $\mathbf{E}_2$ are antiparallel, hence the net field

$$\mathbf{E}=0.$$

Thus *the electric intensity due to two oppositely charged infinite planes is σ/ε_0 at any point in between the planes and is zero for all external points.*

(H) Coulomb's theorem. Consider a charged conductor of irregular shape. In general, surface charge density will vary from point to point. At a small surface δS, let us assume it to be a constant, say σ. Let us construct a gaussian surface in the form of a cylinder AB of cross section δS, one end face of which is inside the conductor and other passes through a point P outside the surface close to it, as shown in Fig. 2.22. The end face B does not contribute to flux as $\mathbf{E}$ is zero everywhere inside the conductor. The curved surface also does not contribute to the flux as $\mathbf{E}$ is always normal to the charged conductor and hence parallel to the curved surface. Thus the only contribution to the flux is through the end face A which is outside the conductor. Thus from Gauss's law

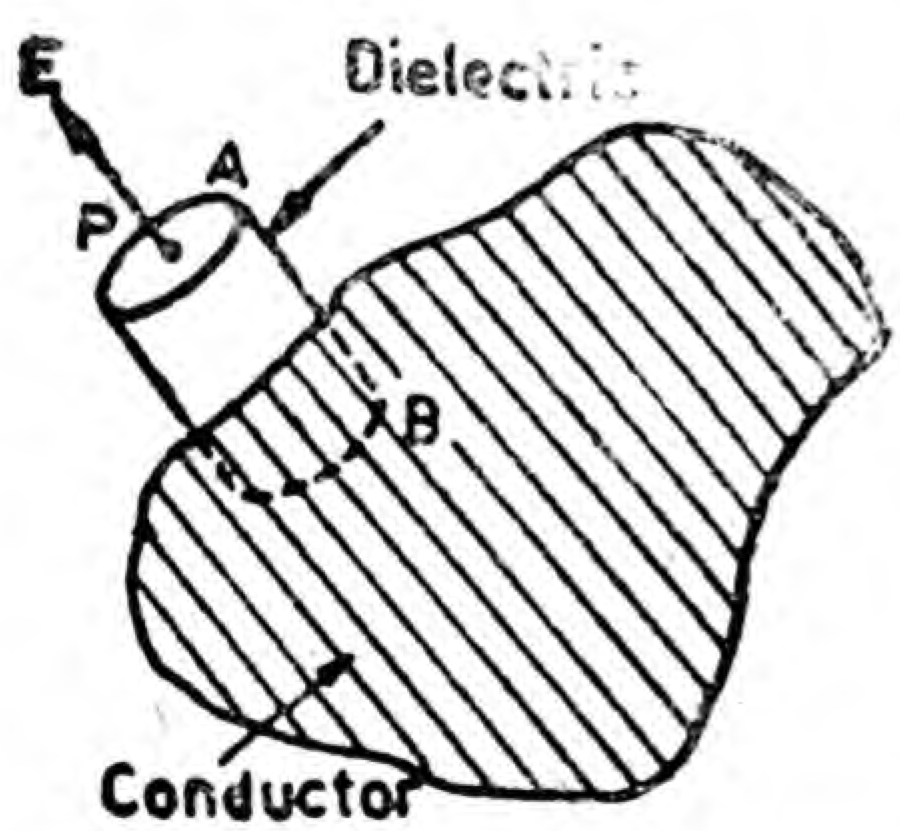


Fig. 2.22.

$$\oint_S \mathbf{E}.d\mathbf{S}=\oint_S EdS=E\delta S=\frac{\sigma\delta S}{\varepsilon_0}$$

or
$$E=\sigma/\epsilon_0. \quad \text{...(54)}$$

This agrees with the results already obtained for spherical, cylindrical and plane surfaces and is known as *Coulomb's theorem.* This theorem states that the electric intensity at any point close to the charged conductor is $1/\varepsilon_0$ times the surface density of charge on the surface.

2.10. ELECTRIC FIELD AND CHARGE IN CONDUCTORS

Remembering that the conductors possess free electrons. If a resultant electric field exists in the conductor, these free charges will experience a force which will set a current flow. When no current flows, the resultant force and the electric field must be zero. Thus, under electrostatic conditions, the value of **E** at all points within a conductor is zero. This idea, together with the Gauss's law can be used to prove several interesting facts.

(*i*) Consider a charged conductor carrying a charge Q and no currents are flowing in it. Now consider a gaussian surface inside the conductor, everywhere on which $\mathbf{E}=0$. Thus from Gauss's law

$$\oint_S \mathbf{E}.d\mathbf{S}=(1/\varepsilon_0)\ \Sigma q,$$

we get

$$\Sigma q=0 \text{ as } E=0. \qquad ...(55)$$

Thus the sum of all charges inside the gaussian surface is zero. This surface can be taken just inside the surface of the conductor, hence any charge on the conductor must be on the surface of the conductor. In other words, *under electrostatic conditions, the excess charge on a conductor resides on its outer surface.*

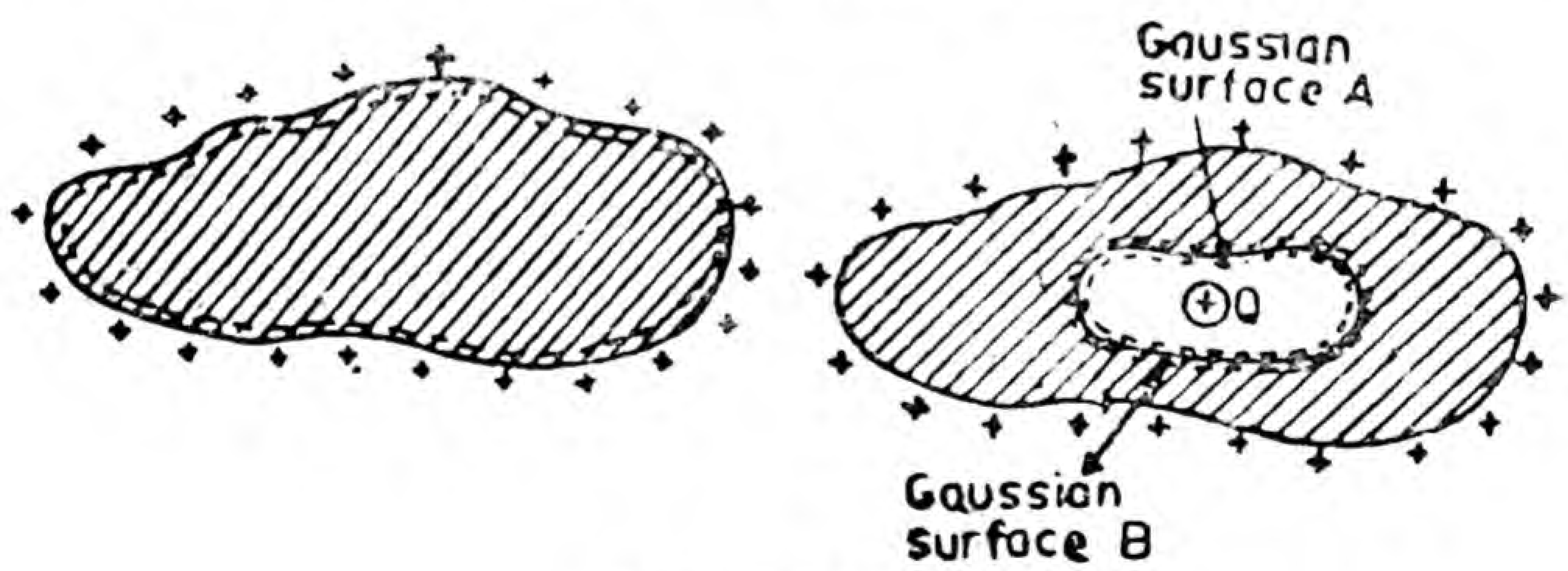


Fig. 2.23.

(*ii*) Consider a charge Q suspended in a cavity in a conductor. Let us apply Gauss's law to the gaussian surface A (within the cavity).

$$\oint_S \mathbf{E}\ .\ d\mathbf{S}=(1/\varepsilon_0)\ \Sigma q=Q/\epsilon_0.$$

Consider a second gaussian surface B, just inside the conductor. $\mathbf{E}=0$ on this gaussian surface, as it is inside the conductor. Hence Gauss's law

$$\oint_S \mathbf{E}\ .\ d\mathbf{S}=(1/\varepsilon_0)\ \Sigma q \text{ gives}$$

$$\Sigma q=0. \qquad ...(56)$$

This concludes that a charge of $-Q$ must reside on the metal surface of the cavity and thus the sum of this induced charge $-Q$ and the original charge Q is zero. In other words, *a charge Q suspended inside a cavity in a conductor induces an equal and opposite charge $-Q$ on the surface of the cavity.*

(*iii*) The same line of approach can be used to show that the field inside the cavity of a conductor is zero when no charge is suspended in it.

(*iv*) As field inside the conductor is zero, the lines of flux coming from Q cannot penetrate into the metal and must terminate on the surface of the cavity. Since lines of flux terminate on negative charge only, hence an equal negative charge must be induced on the surface of the cavity.

2.11 FIELD INSIDE A HOLLOW CONDUCTOR

Let us calculate the field at any point C inside a uniformly charged hollow sphere (spherical shell). The field at point C is obtained by taking the vector sum of all the contributions at this point. Draw through C a double cone of infinitesimal solid angle $\delta\omega$ cutting out areas dS_1 and dS_2 of the spherical shell at distances r_1 and r_2 from point C. From the properties of solid angles, we know that

$$\delta\omega=\alpha_1=\alpha_2=\left(\frac{dS_1\cos\theta_1}{r_1^2}\right)$$

$$=\left(\frac{dS_2\cos\theta_2}{r_2^2}\right)$$

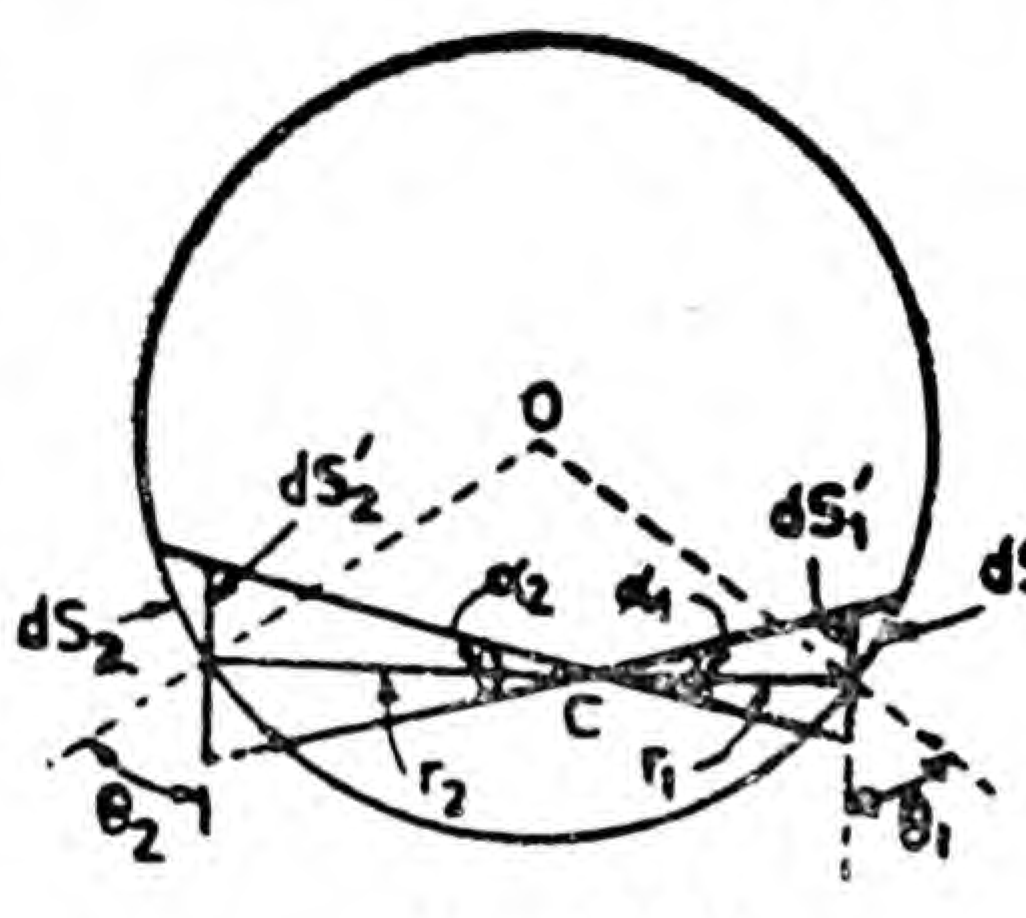


Fig. 2.24.

For small solid angles, the angle between the surface and the right section of the cone, θ_1 and θ_2 are equal, hence

$$\frac{dS_1}{r_1^2}=\frac{dS_2}{r_2^2} \text{ or } \frac{\sigma dS_1}{r_1^2}=\frac{\sigma dS_2}{r_2^2}. \qquad \text{...(57)}$$

If the inverse square law holds, the components of the field at C are

$$\Delta E_1=\frac{1}{4\pi\varepsilon_0}\frac{\sigma dS_1}{r_1^2} \text{ and } \Delta E_2=\frac{1}{4\pi\varepsilon_0}\frac{\sigma dS_2}{r_2^2} \qquad \text{...(58)}$$

and are in opposite directions. Thus we see that ΔE_1 and ΔE_2 are equal in magnitude but opposite in directions, hence the field at point C due to the particular part of the total charge subtended by the pair of equal and opposite elements of solid angles, is zero. Similarly the remaining surface of the sphere can be divided into pairs of elements, hence the electric field at point C due to the whole charge on the shell comes out to be zero.

Thus we see that the resultant electric intensity at this point C is zero, if the inverse square law holds good. Therefore *the inverse square law is proved if electric field inside the charged shell is zero.* This was experimentally shown by *Cavendish.* In his experiment a conducting sphere A was supported inside a bigger concentric sphere B. These were insulated from each other. The outer sphere B was first charged and connected with sphere A with the help of a wire H. The connection between these spheres was broken and the residual charge on the inner sphere was tested by electroscope. No charge was found on the sphere A. Thus he proved that the exponent n in the force (*i.e.* power of r in the relation for electric field) law was two. He repeated his experiment many times and found that n was between 2.02 and 1.98. Cavendish did not publish his results so that almost no body knew about them at that time. Maxwell repeated Cavendish's experiment with more accuracy and obtained values as $2 \pm 5 \times 10^{-5}$. In 1936, Plimpton and Lawton repeated the experiment again and set the limits as $2 \pm 2 \times 10^{-9}$. Thus we see that the inverse square law is correct, but not exactly.

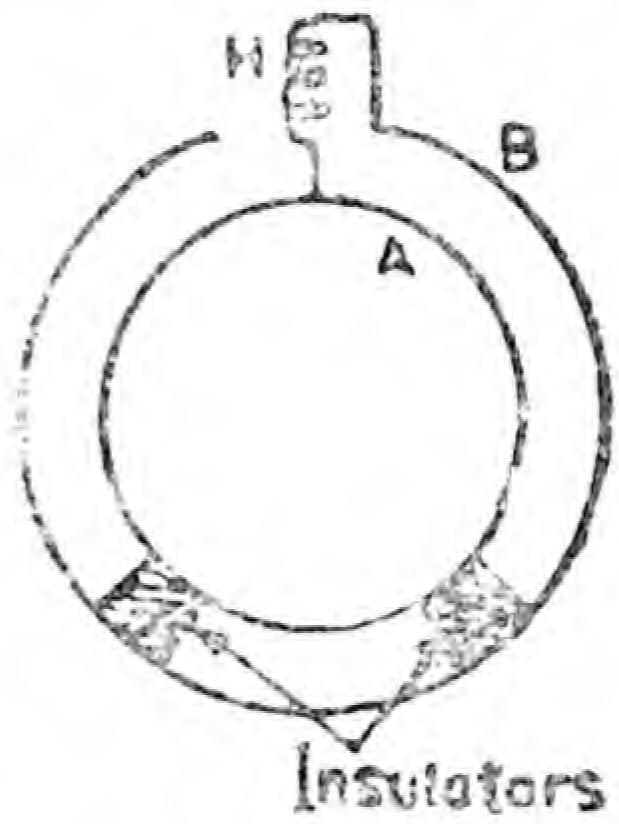


Fig. 2.25.

2.12. MECHANICAL FORCE ON THE CHARGED CONDUCTOR

We know that similar charges repel each other, hence the charge on any part of surface of the conductor is repelled by the charge on its remaining part. The surface of the conductor thus experiences a mechanical force as the charge is bound to the surface. The electric intensity at any point P near the conductor surface can beassumed as due to the small part of the surface of area say δS immediately in the neighbourhood of the point under consideration and due to the rest of the surface. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the electric intensities due to these parts respectively. We know from Coulomb's theorem that the total electric field $\mathbf{E}$ $(=\mathbf{E}_1+\mathbf{E}_2)$ has magnitude σ/ε_0 at any point P just outside the conductor and is zero at point Q just inside the conductor, *i.e.*,

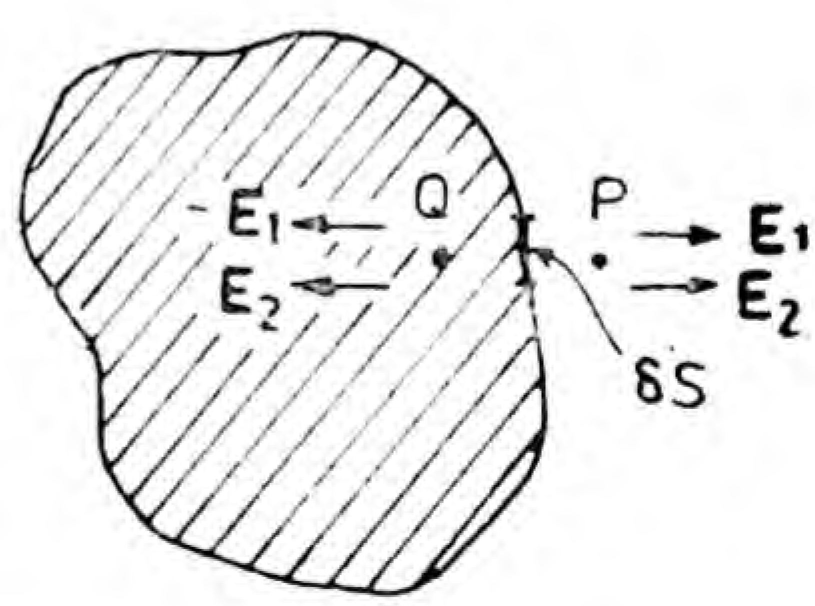


Fig. 2.26.

$$E = E_1 + E_2 = \sigma/\varepsilon_0 \qquad \text{at point } P$$

and
$$-E_1 + E_2 = 0 \qquad \text{at point } Q$$

$$\therefore \qquad E_1 = E_2 = \sigma/2\varepsilon_0.$$

Hence the force experienced by the small surface of area δS due to the charge on the rest of the surface is

$$\sigma \delta S E_2 = \sigma \delta S \ \sigma/2\varepsilon_0 = \sigma^2 \delta S/2\varepsilon_0.$$

Thus the mechanical force per unit area on the charged conductor

$$= \sigma^2/2\varepsilon_0 = (\varepsilon_0 E)^2/2\varepsilon_0 = \varepsilon_0 E^2/2. \qquad ...(59)$$

Its unit is newton/metre^2. It is directed along the outdrawn normal. If the medium is of dielectric constant k, the above relation becomes

Outward mechanical force/area $= \frac{1}{2} \varepsilon_0 k E^2$.

Electrostatic energy in the medium—When a conductor situated in a dielectric is charged, a force acts outward on the surface of the conductor. Imagine the unit area of the charged surface to be moved in the opposite direction through a small distance dx. The work done $dW = \frac{1}{2}\varepsilon_0 k E^2 \times dx$. Since the volume swept out by this imaginary area is dx, hence the work done per unit volume will be $\frac{1}{2} \varepsilon_0 k E^2$. This work is stored as an energy of the medium. Therefore the energy stored up in unit volume of the medium is

$$W = \tfrac{1}{2} \varepsilon_0 k E^2. \text{ joule}/m^3. \qquad ...(60)$$

Let us consider few applications of the mechanical force.

(*i*) **Electrified soap bubble.** We have proved in mechanics that the excess of pressure inwards on a soap bubble of radius r due to surface tension T is given by

$$P = 4T/r.$$

If the bubble is now given a charge q, then the outward mechanical pressure

$$= \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\left(\frac{q}{4\pi r^2}\right)^2 = \frac{q^2}{32\pi^2\varepsilon_0 r^4}.$$

$\therefore$ Net excess of pressure

$$P = \frac{4T}{r} - \frac{q^2}{32\pi^2\varepsilon_0 r^4} = \frac{4}{r}\left[T - \frac{q^2}{128\pi^2\varepsilon_0 r^3}\right]. \qquad ...(61)$$

Thus we see that the effective value of surface tension of the bubble decreases and therefore the bubble will expand. The increase in radius of bubble can be obtained as follows :

We know that $P \propto 1/V$ or $P \propto 1/r^3 \ (= k/r^3)$.

$$\therefore \quad \frac{dP}{dr} = -\frac{3k}{r^4} = -\frac{3P}{r} \quad \text{or} \quad dr = -\left(\frac{r}{3P}\right) dP.$$

As change in pressure due to charge q, $dP=(q^2/32\pi^2\varepsilon_0 r^4)$, hence

$$dr=\frac{r}{3P}\cdot\frac{q^2}{32\pi^2\varepsilon_0 r^4}=\frac{q^2}{96\varepsilon_0\pi^2 Pr^3}. \qquad ...(62)$$

(*ii*) **Formation of a cloud on charged particles.** We have proved in mechanics that the surface tension helps in the contraction of the liquid drop, the inward pressure is $2T/r$. If charge q is given to the drop, the outward mechanical pressure due to this charge is $q^2/32\pi^2\varepsilon_0 r^4$. Thus if the latter is greater than the former, the drop will tend to grow instead of contracting. On the other hand if r is small and the drop is uncharged, *i.e.*, the former is very large, the tendency to form drops on uncharged particles is extremely small. But when there is a charged particle present in saturated vapour which is suddenly cooled, drops of finite size will be formed. Thus a dust free super-saturated vapour may be used to detect the presence of charged particles. This principle was used by C.T.R. Wilson for the detection of radioactive particles. The discussion of the apparatus used by Wilson, known as Cloud Chamber is beyond the limit of this book,

2.13. DETERMINATION OF ELECTRONIC CHARGE

The first attempt to determine the electronic charge was made by J.J. Thomson in 1898. He used a method developed by C.T.R. Wilson and later on modified by H.A. Wilson. The method was based on the principle that the cloud of water droplets could be formed on charged ions under special conditions. This cloud falls down with a certain velocity. Its velocity was observed and the value of '*e*' was determined. The value so obtained was 40% higher than the standard value determined later.

In 1909, R.A. Millikan not only demonstrated the discrete nature of electric charge but gave the correct value of the charge on an individual electron. In this experiment two actually parallel horizontal metal plates *A* and *B* are insulated from each

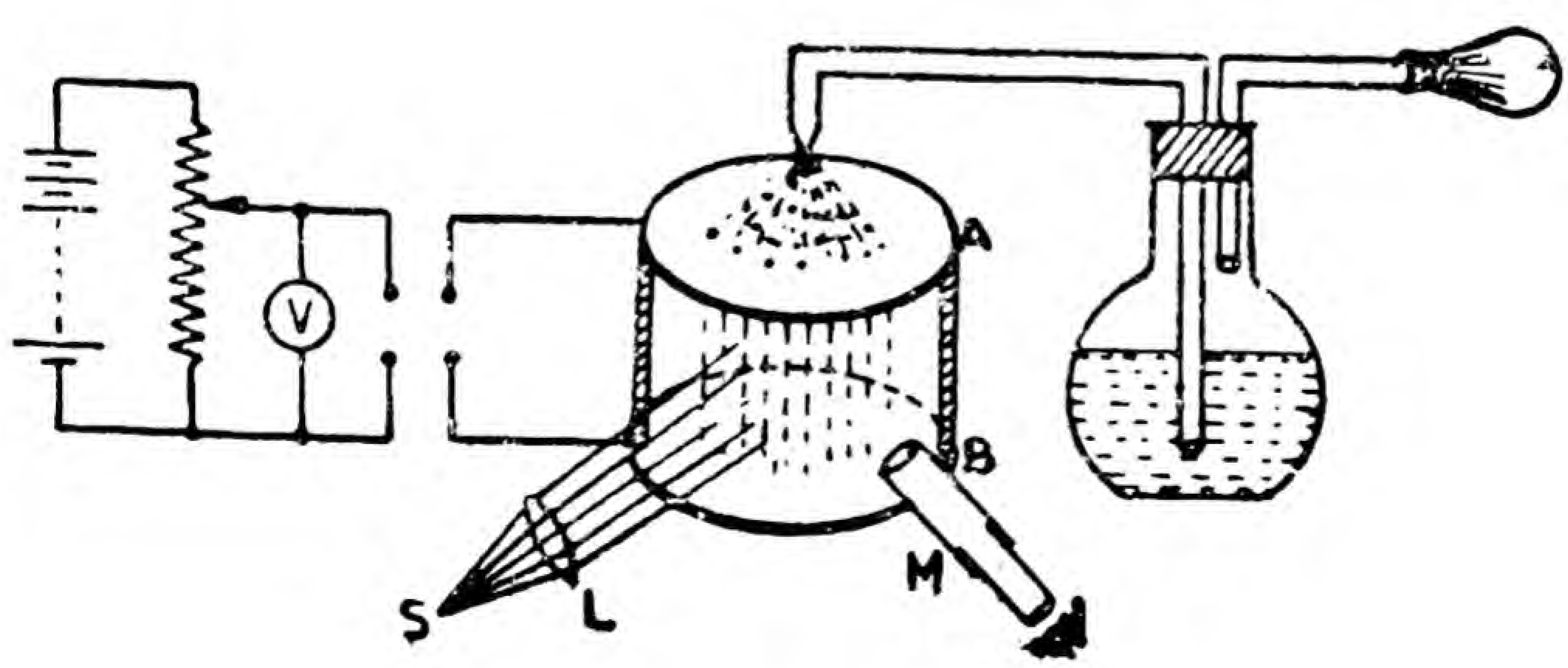


Fig. 2.27.

other and separated by ebonite or glass pieces of thickness about a cm. Potential difference is applied between these plates by connecting a power supply through a reversing switch with the help of which the direction of electric current can be changed or can be set zero. The upper plate *A* has some fine holes, through which few of the droplets of a non-volatile oil from an atomiser are allowed to fall. A majority of these drops becomes charged by friction at the atomiser nozzle during the time of spray. The droplets, illuminated by the light beam, appear as tiny bright stars falling slowly with a terminal velocity governed by their weights, viscous force and by buoyant force. These droplets are observed through a microscope *M* whose eyepiece carries a micrometer scale at the position of cross wires to measure the distance travelled by the droplets.

When electric field is not applied, a drop falls down under gravity, opposed by the viscous force and bouyancy force and acquires a steady terminal velocity v. If a be the radius of the oil drop, ρ the density of oil and σ the density of air then by Stokes law, we get

Viscous force $6\pi\eta a v$=Apparent weight of the drop

$$=\tfrac{4}{3}\pi a^3\rho g-\tfrac{4}{3}\pi a^3\sigma g$$

or
$$a=[9\eta v/2(\rho-\sigma)g]^{1/2}. \qquad \ldots(63)$$

If the charge on the droplet is n times the electronic charge, denoted by e, and the plates are maintained at a potential difference V such that E $(=V/d)$ is in downward direction. The force on the—vely charged drop will be in upward direction and having value neE or neV/d. On the other hand the downward velocity of+vely charged droplets will increase further and the velocity of uncharged particles remains same. If one of these droplets is selected for observations. Firstly this droplet is allowed to move under the electric field. This field is so chosen that the droplet moves with a constant velocity, known as terminal velocity. The velocity v' is measured with the help of a stop-watch. In equilibrium, downward forces=upward forces, *i.e.*,

$$\tfrac{4}{3}\pi a^3\rho g=\tfrac{4}{3}\pi a^3\sigma g+6\pi\eta a v'+neE$$

$$neE=\tfrac{4}{3}\pi a^3(\rho-\sigma)\,g-6\pi\eta a v'=6\pi\eta a[v-v']$$

or
$$ne=9\sqrt{2}\pi\eta^{3/2}\,(\rho-\sigma)^{-1/2}\,g^{-1/2}\,v^{1/2}(v-v')/E$$

$$=9\pi\left[\frac{2\eta^3}{g(\rho-\sigma)}\right]^{1/2}\frac{v^{1/2}}{E}\,(v-v'). \qquad \ldots(64)$$

The value of ne is calculated for each set separately by observing velocities of the droplets in the presence and in the absence of an electric field. These values of charges n_1e, n_2e,...are divided by the highest common factor of all such values. Milikan measured the charges of some thousands of drops and found that the charge on any one droplet was integral multiple of a basic charge e. He gave the value of e as 1.60207×10^{-19} coulomb.

Exercises

Example 1. *Prove that the gravitational force is negligible in comparison to electrostatic force in the hydrogen atom in which the electron and the proton are about 5.3×10^{-11} meter apart.*

From Coulomb's law, the electrostatic force between electron and proton

$$F_e = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}$$

$$= (9.0 \times 10^9 \text{ newton. m}^2/\text{coul}^2) \frac{(1.6 \times 10^{-19} \text{ coul})^2}{(5.3 \times 10^{-11} \text{ meter})^2}$$

$$= 8.2 \times 10^{-8} \text{ newton.}$$

From gravitational law, the gravitational force

$$F_g = G\frac{m_1 m_2}{r^2}$$

$$= (6.7 \times 10^{-11}\ N.\ m^2/kg^2) \frac{(9.1 \times 10^{-31}\ kg)(1.7 \times 10^{-27}\ kg)}{(5.3 \times 10^{-11} \text{ meter})^2}$$

$$= 3.69 \times 10^{-47} \text{ newton.}$$

which is about 10^{-39} times smaller than the electrostatic force, thus can be neglected in comparison to the latter in nuclear or atomic problems.

Example 2. *Two pitch balls, each weight 100 mg and suspened from the same point by silk threads, 30 cms. long, are equally charged and repel each other to a distance of 10 cms. What is the charge on each ball ?*

Let two pitch balls each having a charge q are suspended from the same point P. Due to repulsive force they are at a distance r apart. Each ball is acted upon by following forces :

(*a*) Its weight acting vertically downward

$$mg = 100 \times 10^{-6} \text{ kg.} \times 9.8 \text{ meter/sec}^2$$
$$= 9.8 \times 10^{-4} \text{ newton.}$$

(*b*) Force of repulsion acting along **F**

$$F = \frac{q^2}{4\pi\varepsilon_0\, r^2}.$$

$$= \frac{9 \times 10^9 \text{ nt m}^2}{\text{coul}^2} \times \frac{q^2}{(0.1 \text{ metre})^2}$$

$$= 9 \times 10^{11}\ q^2 \text{ newton.}$$

(*c*) The tension T of the string acting along the string, along **T**.

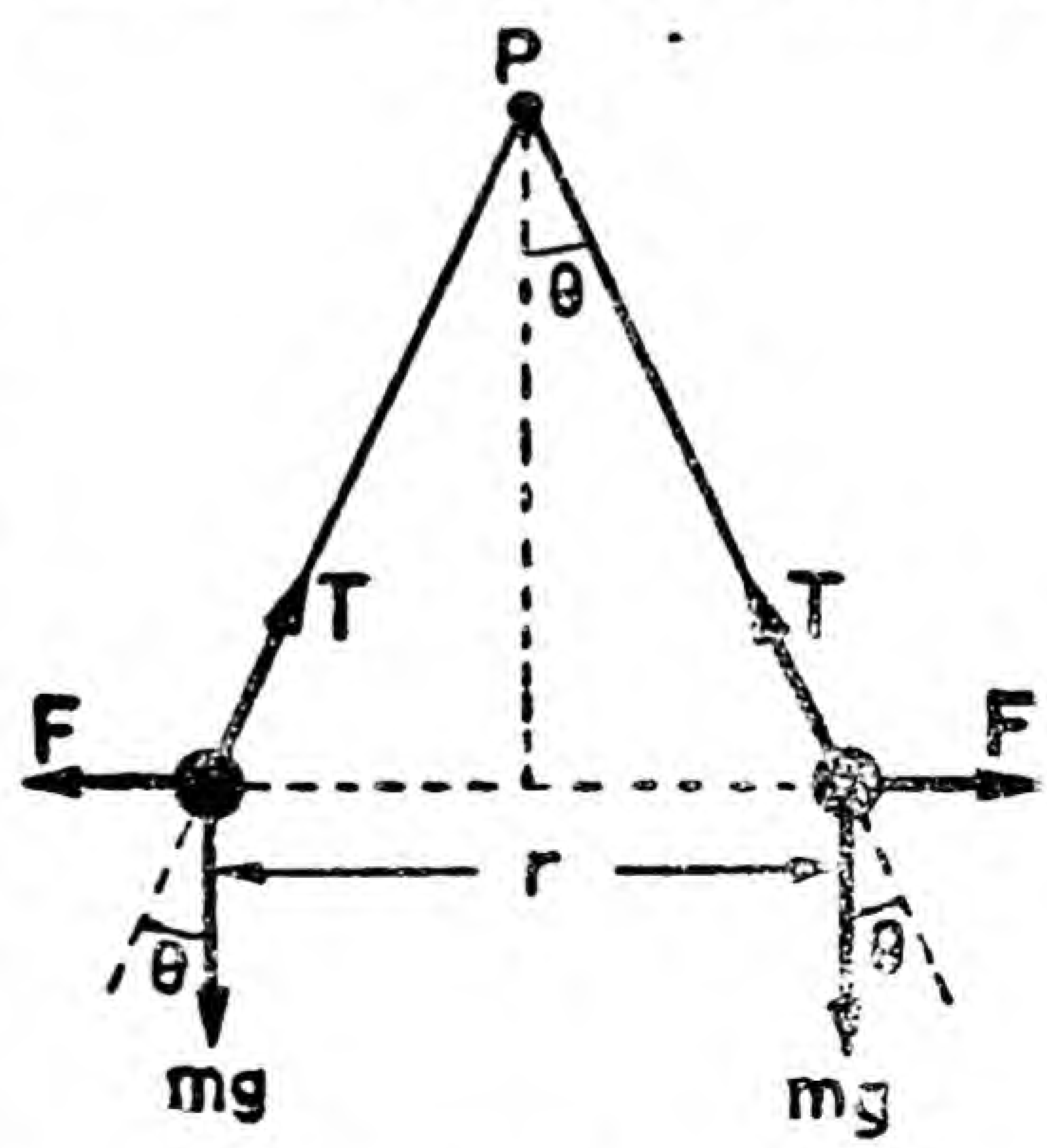


Fig. 2.28

In equilibrium, moment of these forces about any point say point of suspension P will be zero, hence

$$mg\,(r/2)\sin\theta = F(r/2)\cos\theta \quad \text{or} \quad F = mg\tan\theta$$

$$\therefore \quad 9\times10^{11}\,q^2 = 9.8\times10^{-4}\,\frac{5\times10^{-2}}{(30^2-5^2)^{1/2}\times10^{-2}}$$

or

$$q^2 = 1.841\times10^{-16} \text{ and } q = 1.375\times10^{-8} \text{ coulomb.}$$

Example 3. *ABCD is a square of 4 cm side, charges of $16\times10^{-9}, -16\times10^{-9}$ and 32×10^{-9} coulomb are placed at the points A, C and D respectively. Find the intensity of the field at point B.*

The electric field intensity $\mathbf{E}$ due to a point charge q at a distance r is given by the relation.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2}\,\hat{r},$$

where r is along r increasing.

As $\mathbf{E}$ is a vector quantity, the total intensity at B is the vector sum of three intensities due to charges at A, C and D, where

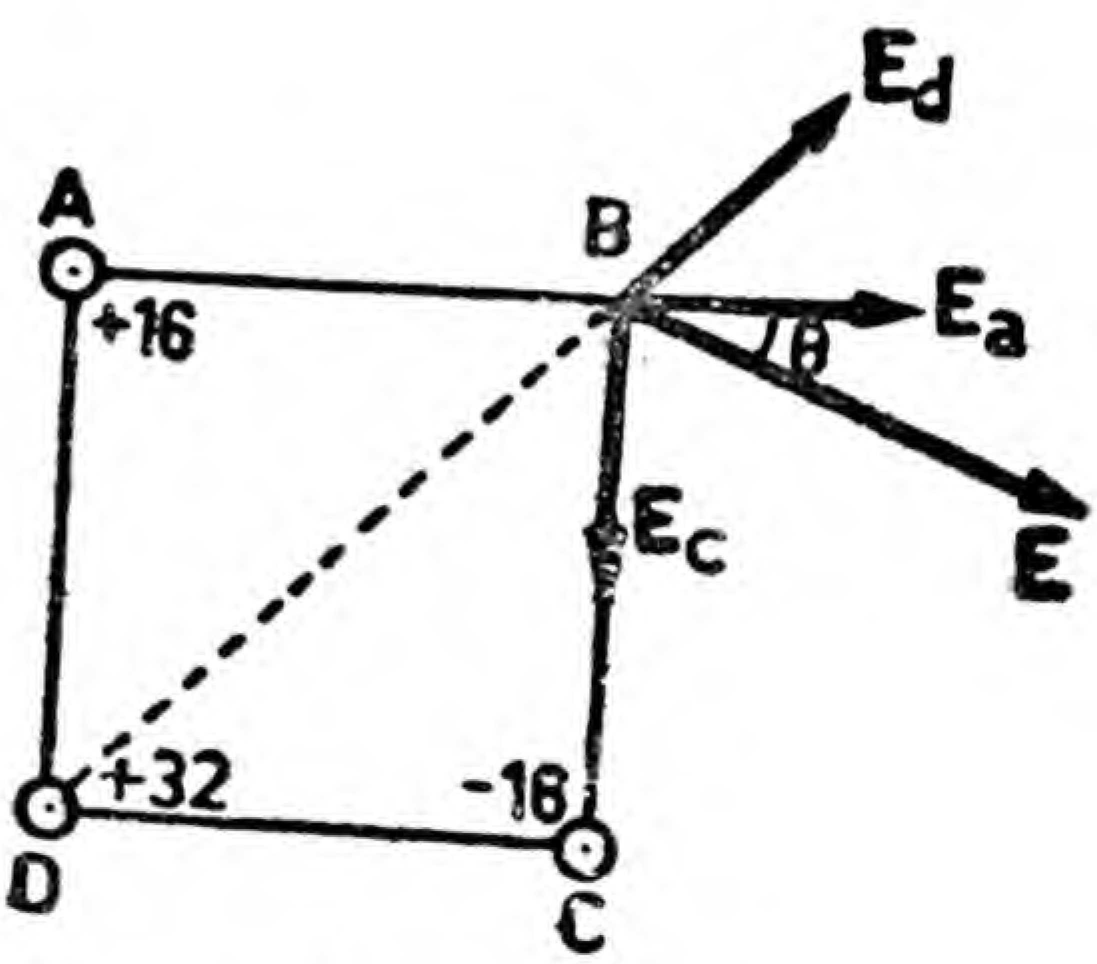


Fig. 2.29.

$$E_a = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r^2}$$

$$= 9\times10^9 \times \frac{16\times10^{-9}}{(0.04)^2} = 9\times10^4 \text{ along AB}$$

$$E_c = 9\times10^9 \times \frac{(-16\times10^{-9})}{(0.04)^2} = 9\times10^4 \text{ along BC,}$$

and

$$E_d = 9\times10^9 \times \frac{32\times10^{-9}}{32\times10^{-4}} = 9\times10^4 \text{ along DB.}$$

Resultant intensity $\mathbf{E} = \mathbf{E}_a + \mathbf{E}_c + \mathbf{E}_d$.

$$\therefore \quad E = \sqrt{(E_x^2+E_y^2)} = \sqrt{[(E_a+E_d\cos45^\circ)^2+(E_c-E_d\sin45^\circ)^2]}$$

$$= \sqrt{[E_a^2+E_d^2+E_c^2+2E_d(E_a\cos45^\circ - E_c\sin45^\circ)]}$$

$$= 9\sqrt{3}\times10^4 \text{ newtons/coul.}$$

and $\tan\theta = \frac{E_y}{E_x} = \frac{E_c - E_d \sin 45°}{E_a + E_d \cos 45°} = \frac{(1-1/\sqrt{2})9\times10^4}{(1+1/\sqrt{2})9\times10^4}$

$$=0.171$$

or $\theta = 9°\ 44'$.

Example 4. *An electron is constrained to move along the axis of the ring of charge q and of radius a. Show that the electron can perform oscillations whose frequency is given by*

$$\omega = \sqrt{eq/4\pi\varepsilon_0\, ma^3}.$$

Electric intensity **E** at a point *P* on the axis of the ring of radius *a* at a distance *x* from its centre, as given in article 2.4 (*c*), is given by

$$E = \frac{1}{4\pi\varepsilon_0}\,\frac{qx}{(a^2+x^2)^{3/2}}. \qquad \ldots(26)$$

It is along the axis of the ring.

The force on the electron if placed at point *P*

$$F = qex/4\pi\varepsilon_0\,(a^2+x^2)^{3/2}.$$

If $x \ll a$, then the force is proportional to the distance from the mean position *i e*, the centre of the ring. It means that the electron executes simple harmonic motion about the centre of the ring. The frequency is given by the relation

$$\omega = \sqrt{\frac{d^2x/dt}{x}} = \sqrt{\frac{F/m}{x}} = \sqrt{\frac{q\,e}{4\pi\varepsilon_0 ma^3}}.$$

Example 5. *Show the variation of electric field due to the positive charge in the gold atom (Z = 79).*

Gold atom is spherical and is of radius 1×10^{-10} meter. The positive charge of the atom is $79\times1.6\times10^{-19}$ coul. and is distributed uniformly within the nucleus of radius 6.9×10^{-15} meter. The electric intensity outside the nucleus is given by the relation (40), *i.e.*,

$$E = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r^2}.$$

At the surface of atom $r = 10^{-10}$ meter

$$E = \frac{(9\times10^9 \text{ nt. m}^2/\text{coul}^2)\ (79\times1.6\times10^{-19} \text{ coul})}{(10^{-10} \text{ meter})^2}$$

$$=1.14\times10^{13} \text{ nt/coul.}$$

At the nuclear surface $r=R=6.9\times10^{-15}$ meter and

$$E=\frac{(9\times10^{9}\text{ nt. m}^2/\text{coul}^2)\ (79\times1.6\times10^{-19}\text{ coul}}{(6.9\times10^{-15}\text{ meter})^2}$$

$$=2.39\times10^{21}\text{ nt./coul.}$$

Inside the nucleus the electric intensity is given by the relation

$$E=\frac{1}{4\pi\varepsilon_0}\ \frac{q}{R^3}\ r.$$

It is zero at the centre ($r=0$) and increases with r upto the maximum value 2.39×10^{21} nt./coul. at nuclear surface. Beyond it, decreases inversely as the square of the distance from the centre.

Example 6. *If the electric field is given by* $\vec{E}=8\hat{x}+4\hat{y}+3\hat{z}$, *calculate the electric flux through a surface of area 100 units lying in the x-y plane.*

In this problem $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors along x, y and z axes respectively, generally taken as **i**, **j** and **k**.

Since the electric flux is defined as $\Phi=\mathbf{E}.\mathbf{S}$. The vector surface area in the x-y plane will be in the direction of the out drawn normal, *i.e.*, the z-direction. Therefore

$$\mathbf{S}=100\ \hat{z}(=100\ \mathbf{k})$$

$$\Phi=\mathbf{E}\text{-}\mathbf{S}.=(8\hat{x}+4\hat{y}+3\hat{z}).\ 100\ \hat{z}$$

$$=300\text{ units.}$$

Example 7. *Find the electric field at the centre of a uniformly charged semicircular arc.*

Let the charge per unit length of arc be λ and the radius of the arc be a. If the arc is divided into small segments, so that each will work as a point charge. The electric field at point P due to a segment of length ΔL,

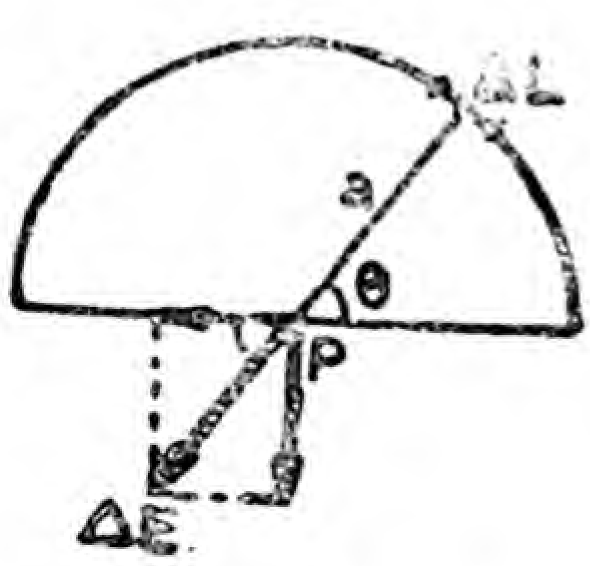


Fig. 2.30.

$$\Delta E=\frac{1}{4\pi\varepsilon_0}\ \frac{\lambda\Delta L}{a^2}\ .$$

Its x—component $\Delta E_x=\Delta E\cos\theta$ will be cancelled by the contribution from a symmetrically placed ΔL on the left half of the arc, hence net component $E_x=\Sigma\Delta E_x=0$. Thus the y-component contributes to E only.

$$\therefore \quad E=E_y=\sum \Delta E_y=\int \frac{\lambda}{4\pi\varepsilon_0 a^2} \cdot dl \sin \theta.$$

To convert two variables l and θ in one, let us use the relation $dl=a\,d\theta$ (arc=radius × angle subtended by the arc at the centre).

$$\therefore \quad E=\frac{\lambda}{4\pi\varepsilon_0 a}\int_{-\pi}^{\pi} \sin\theta\, d\theta.$$

$$=\lambda/2\pi\varepsilon_0 a.$$

Example 8. *A delectric cylinder of radius a is infinitely long. It is non-uniformly charged such that volume charge density* ρ *varies directly as the distance from the cylinder. Calculate the electric intensity due to it, if* ρ *is zero at the axis and is* ρ_s *on the surface.*

To calculate net charge within the charged cylinder, let us divide it in large number of co-axial thin cylindrical shells. Consider a co-axial shell of radii x and $x+\delta x$ and of length l. If the thickness δx is very small so that the charge density can be assumed as constant at each point on it. Thus the charge on this shell

$$\delta q=2\pi x l\, \delta x\, \rho.$$

As ρ varies directly as the distance x, hence ρ can be written as cx, where c is a constant given by the relation

$$ca=\rho_s \text{ or } c=\rho_s/a.$$

The intensity at any point P can be calculated as given in article 2.9 (*d*). The field E at the point at a distance r outside the cylinder is given by

$$E\, 2\pi r l=\frac{q}{\varepsilon_0}=\frac{1}{\varepsilon_0}\int dq.$$

Charge inside the gaussian surface is the charge on the cylinder of length l, *i.e.*, the integral of dq must be over the whole cylinder.

$$\therefore \quad E=\frac{1}{2\pi r l\varepsilon_0}\int_0^a 2\pi l c x^2\, dx=\frac{ca^3}{3r\varepsilon_0}=\frac{\rho_s a^2}{3r\varepsilon_0}.$$

At points inside the cylinder, E is given by

$$E\, 2\pi r l=\frac{q'}{\varepsilon_0}=\frac{1}{\varepsilon_0}\int dq=\frac{1}{\varepsilon_0}\int_0^r 2\pi l c x^2\, dx,$$

as the charge inside the gaussian surface is the charge on the cylinder of radius r and of length l. Therefore

$$E=cr^2/3\varepsilon_0=\rho_s r^2/3a\varepsilon_0.$$

At the points on the surface of the cylinder, $E=\rho_s a/3\varepsilon_0$

In all the cases the field is in radially outward direction.

Example 9. *Calculate the electric field intensity due to a spherical charge distribution, given by*

$$\rho=\rho_0(1-r/a), \text{ when } r\leqslant a$$

and

$$\rho=0 \qquad \text{when } r>a.$$

Find the value of r for which **E** *is maximum.*

Let us consider a thin spherical shell of radii x and $x+dx$. The volume of this shell $\delta V=4\pi x^2 dx$. To calculate electric field intensity let us assume a gaussian spherical surface passing through the point which is at a distance r from the centre. Using Gauss's law, we get

$$\int_S \mathbf{E}.\, d\mathbf{S}=\frac{1}{\varepsilon_0}\int dq=\frac{1}{\varepsilon_0}\int_0^a \rho_0\left(1-\frac{x}{a}\right)4\pi x^2 dx.$$

Here integration is between $x=0$ to $x=a$, as the whole charge is within the gaussian surface.

$$\therefore \qquad \oint EdS=4\pi r^2\, E=\pi\rho_0 a^3/3\varepsilon_0$$

or

$$E=\rho_0 a^3/12\varepsilon_0 r^2.$$

Field inside the charge distribution is given by

$$\int_S \mathbf{E}.\, d\mathbf{S}=\frac{1}{\varepsilon_0}\int dq=\frac{1}{\varepsilon_0}\int_0^r \rho_0\left(1-\frac{x}{a}\right)4\pi x^2 dx.$$

Here integration is between $x=0$ to $x=r$, as the charge inside the gaussian surface is the charge within the sphere of radius r

$$\therefore \qquad \oint EdS=4\pi r^2\, E=\frac{4\pi\rho_0}{\varepsilon_0}\left[\frac{r^3}{3}-\frac{r^4}{4a}\right],$$

or

$$E=\frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3}-\frac{r^2}{4a}\right].$$

Electric field E will be maximum, when

$$\frac{dE}{dr}=0, \quad \text{or} \quad \frac{d}{dr}\left[\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3}-\frac{r^2}{4a}\right)\right]=0.$$

$$\therefore \quad r=2a/3.$$

Example 10. *If a heavy atom can be pictured as a spherical nucleus with charge $+q$ and radius a embeded in a much larger sphere of negative charge, the electrons. This negative charge is distributed uniformly throughout this sphere of radius b. Find the electric field intensity outside the nucleus.*

The total charge inside the gaussian surface passing through any point *outside the atom* is zero, hence by Gauss's law

$$\int \mathbf{E}.d\mathbf{S}=(1/\varepsilon_0)\Sigma q=0.$$

As surface area $d\mathbf{S}$ is always $\parallel$ to the radius vector, hence $\mathbf{E}$ cannot be $\perp$ to $d\mathbf{S}$ and thus zero.

At a point inside the atom but outside the nucleus and a distance r apart from the centre of the nucleus, the total charge within the gaussian sphere is $\Sigma q=q-(\frac{4}{3}\pi r^3)\,\rho$.

$$\therefore \quad \int \mathbf{E}.d\mathbf{S}=\frac{1}{\varepsilon_0}\left[q-\frac{4}{3}\pi r^3.\left(\frac{q}{\frac{4}{3}\pi b^3}\right)\right]$$

$$E4\pi r^2=\frac{q}{\epsilon_0}\left[1-\left(\frac{r}{b}\right)^3\right] \quad \text{or} \quad E=\frac{q}{4\pi\epsilon_0 r^2}\left[1-\left(\frac{r}{b}\right)^3\right].$$

The first term is due to the nucleus, while the second term represents the cancelling field due to electrons. At point very close to nucleus the second term is negligible, the nucleus and innermost electrons of an atom are therefore not greatly affected by the outer electrons of the atom.

Example 11. *Two plane metal plates are placed parallel to each other, one carries a surface charge density $+\sigma$ and the other -2σ. Find the charge densities on the two surfaces of the third plane metal plate placed at the centre. The central plate is parallel to the other plates and is assumed connected to the earth.*

The lines of force coming from the plate on the left must end on the left side of the central plate. Thus the charge density must be $-\sigma$. Similarly the charge density on the right surface of the central plate must be $+2\sigma$, as it is due to the lines of force coming from the right plate having charge density -2σ. Thus the central plate has a net charge $+\sigma$.

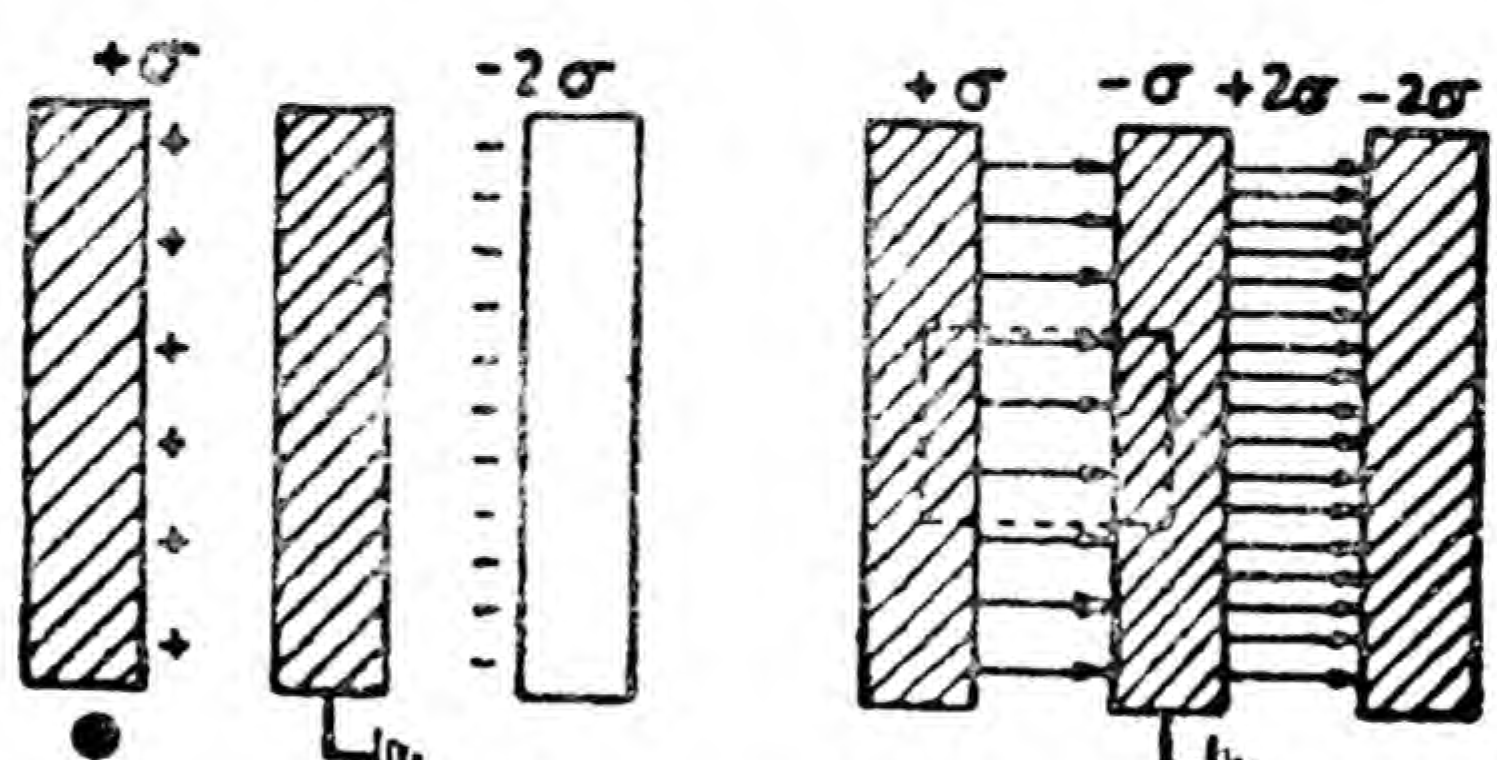


Fig. 2.31

The same result can be obtained by using Gauss's law. Consider a gaussian surface shown by the dotted lines in Fig. 2.31. Thus by Gauss's law.

$$\int \mathbf{E}.d\mathbf{S}=\frac{1}{\varepsilon_0}\Sigma q.$$

As the net flux coming out of the gaussian surface is zero, hence Σq must be zero. This can be true only if the charge on the left side of the central plate is equal and opposite to the charge on the left plate. Similarly we can show that the charge on the right side will be equal and opposite to the charge on the right plate.

Example 12. *Calculate the flux of the electric field strength through each of the faces of a closed cube of length l, if a charge q is placed at its centre and at one of its vertices.*

Symmetry of six faces of a cube about its centre ensures that the flux Φ of **E** through each of the faces will be same when charge q is placed at the centre.

$$\therefore \quad \Phi = 6\Phi s = q/\varepsilon_0 \text{ or } \Phi s = q/6\varepsilon_0.$$

In the second case, when q is at one of the vertices, the flux through each of the three faces meeting at this vertex will be zero, as **E** is parallel to these surfaces. The flux through other three faces will be same, say Φs. We know that one-eighth of the flux originating from charge q passes through this cube, as the faces of the cube meeting at the charge include one-eighth of the spherical surface drawn with charge q as centre.

$$\therefore \quad 3\Phi s = \frac{1}{8}\frac{q}{\varepsilon_0}, \text{ or } \Phi s = \frac{1}{24}\cdot\frac{q}{\varepsilon_0}.$$

Example 13. *Calculate the flux of* **E** *through (i) each of the bases, (ii) the curved surface of a right cylindrical closed surface of radius a and length l, due to the charge q situated at its geometrical centre.*

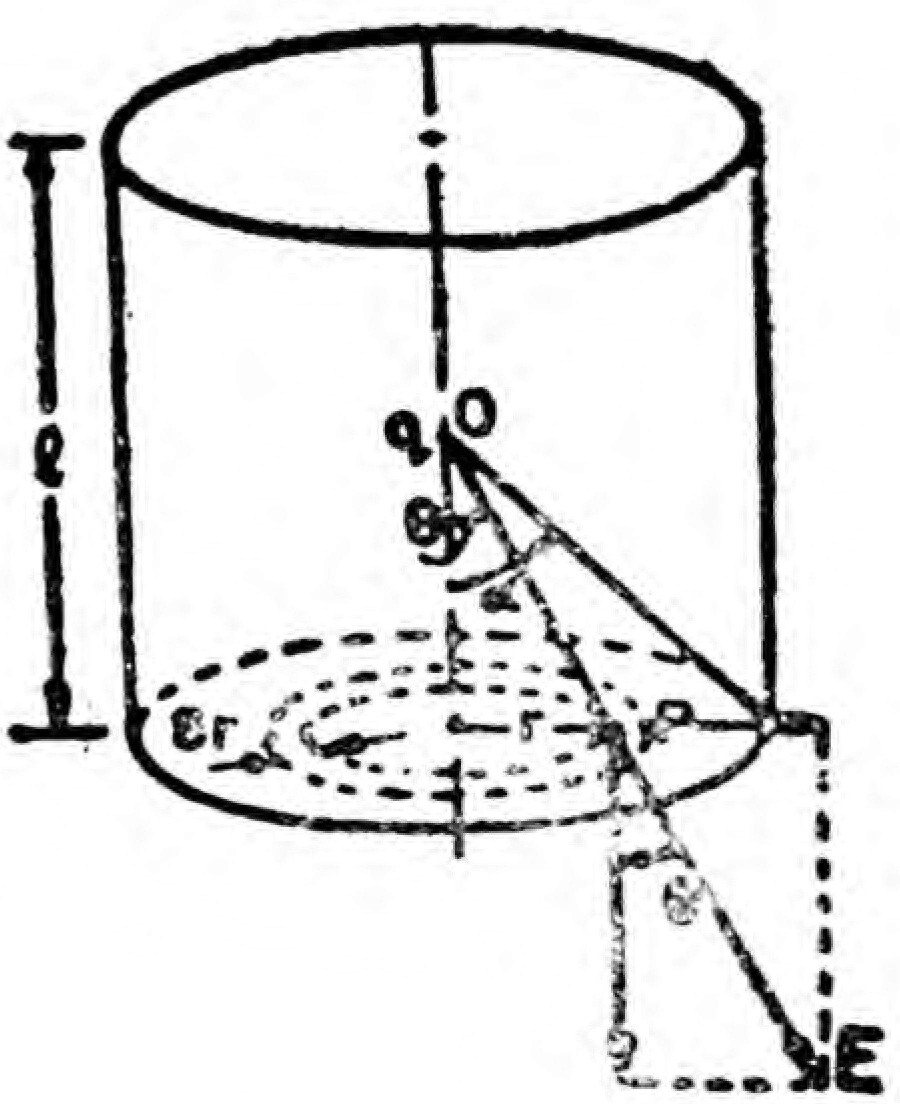


Fig. 2.32

To find the electric flux through the base of the cylinner, let us divide the base (say lower) into large number of concentricr ings. The electric field at any point P of an annular ring between radii r and $r+\delta r$ is

$$E = q/4\pi\varepsilon_0(r^2+l^2/4) \text{ along } \mathbf{OP}.$$

The flux through a very small element of this annular ring is $E\cos\theta \times$ area of the element. As $E\cos\theta$ is same for all elements on this annular ring, hence the flux through this annular ring

$$\delta\Phi = E\cos\theta . 2\pi r\delta r$$

$$= \frac{q.2\pi r\delta r \cos\theta}{4\pi\varepsilon_0(r^2+l^2/4)} = \frac{q l r\delta r}{4\varepsilon_0(r^2+l^2/4)^{3/2}}$$

∴ Total flux through the base of the cylinder

$$\Phi=\int_0^a \frac{qlrdr}{4\varepsilon_0(r^2+l^2/4)^{3/2}}=\frac{q}{2\varepsilon_0}\left[1-\frac{l/2}{(a^2+l^2/4)^{1/2}}\right].$$

By symmetry the flux through the above base will also be same. As the total flux originating from a point charge is always q/ε_0, hence by Gauss's law

$$\Phi_{curved}+2\Phi_{bases}=q/\varepsilon_0$$

or $$\Phi_{curved}=\frac{q}{\varepsilon_0}-\frac{q}{\varepsilon_0}\left[1-\frac{l/2}{(a^2+l^2/4)^{1/2}}\right]=\frac{ql}{2\varepsilon_0(a^2+l^2/4)^{1/2}}.$$

Example 14. *Calculate electric flux for a cube of side a as shown in Fig. 2.33, where $E_x=bx^{1/2}$, $E_y=E_z=0$, $a=10$ cm and $b=800$ nt/coul-m$^{1/2}$.*

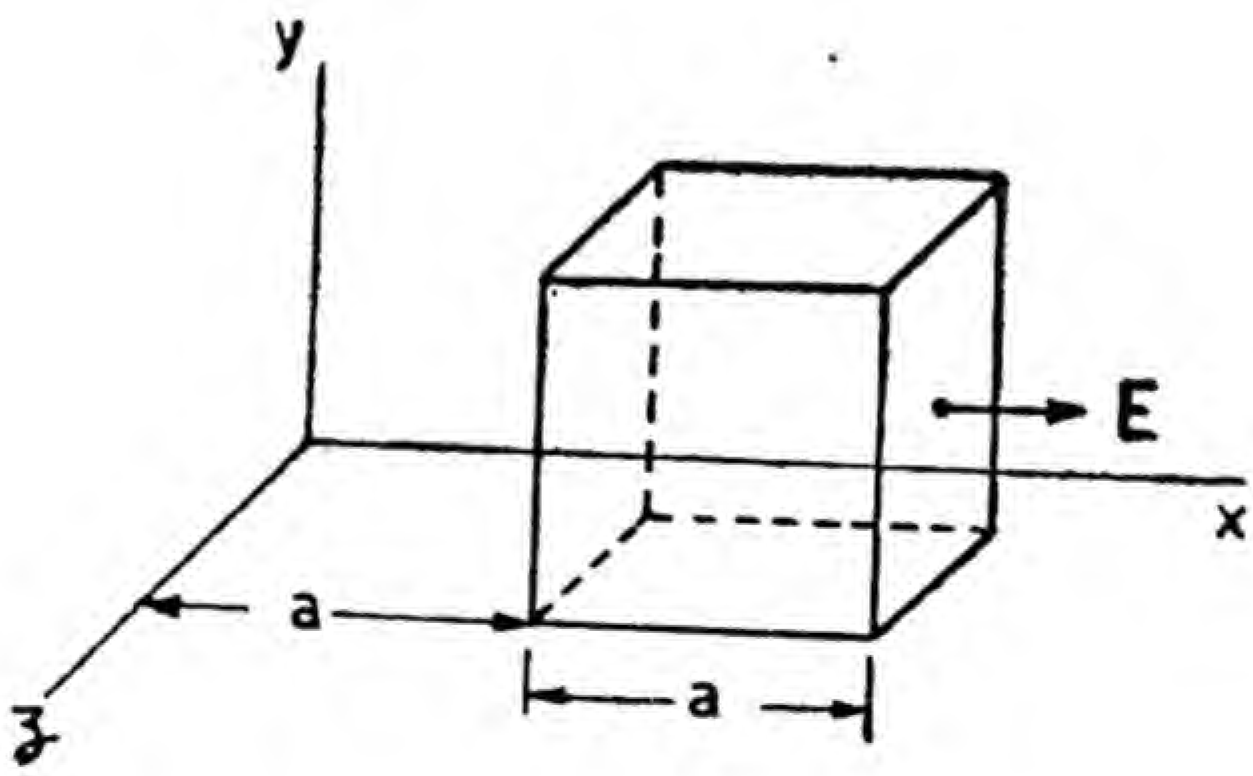


Fig. 2.33.

Since the electric field is acting only in x-direction and its y- and z-components are zero. The net electric flux is thus given as

$$\Phi=\Phi_{right\ surface}-\Phi_{left\ surface}$$
$$=\int \mathbf{E}.d\mathbf{S}-\int \mathbf{E}.d\mathbf{S}$$
$$=(E_x.a^2)_{right}-(E_x a^2)_{left}=a^2[E_{2a}-E_a]$$
$$=a^2[b\,(2a)^{1/2}-ba^{1/2}]$$
$$=ba^{5/2}\,(\sqrt{2}-1)=800\times(0.1)^{5/2}\,(\sqrt{2}-1)$$
$$=1.05 \text{ mks units.}$$

Example 15. *If the electric field near the earth's surface be 300 volts/meter directed downwards, what is the surface density of charge on the earth's surface ?*

The electric field just outside a charged conductor of surface charge density σ is given by

$$E=\sigma/\varepsilon_0.$$

In this problem the electric field near the earth's surface, which may be treated as a charged conductor, is given as 300 volts/meter. Since the permittivity of the free space $\varepsilon_0=8.85\times10^{-12}$ coul^2/nt-m^2. Thus the surface density of charge on the earth's surface

$$\sigma=\varepsilon_0 E=8.85\times10^{-12}\times300=2.66\times10^{-9} \text{ coul/m}^2.$$

Example 16. *A gold foil weighing 50 mg per sq cm is placed on a horizontal charged plate. What would be the density of charge so that the foil may just rise ?*

Mechanical force per unit area of a charged plate$=\sigma^2/2\varepsilon_0$.

In this problem outward mechanical force is balanced by the force due to the weight of gold foil. Hence at equilibrium

$$\sigma^2/2\varepsilon_0=mg, \quad \text{or} \quad \sigma^2=2\varepsilon_0\, mg,$$

where σ is the surface charge density and m is the mass per unit area of the foil.

$$\therefore \quad \sigma^2=2\times8.85\times10^{-12}\times50\times10^{-2}\times9.8$$

or $$\sigma=9.313\times10^{-6} \text{ coul/m}^2.$$

Example 17. *Derive an expression for the electric charge required to expand the bubble to twice its dimensions.*

Let P_1 be the internal pressure inside the bubble of radius r and surface tension T. If P be the atmospheric pressure, then

$$P_1=P+4T/r.$$

When the bubble after charging expands, the radius becomes $2r$, the volume becomes 8 times the initial volume. Therefore by Boyle's law the pressure inside the bubble reduces to $P_1/8$. This decrease in pressure is due to the mechanical outward pressure due to the charge q given to the bubble.

$\therefore$ Total outward pressure=total inward pressure

$$\frac{1}{8}\left(P+\frac{4T}{r}\right)+\frac{q^2}{32\varepsilon_0\pi^2(2r)^4}=P+\frac{4T}{2r}$$

$$q^2=64\pi^2\varepsilon_0\, r^3[7Pr+12T]$$

or $$q=8\pi\sqrt{\varepsilon_0 r^3(7Pr+12T)}.$$

Example 18. *An insulated soap bubble 10 cm in radius is charged with 20 stat-coulomb. Find the increase in radius due to the charge. Given the atmospheric pressure$=10^5$nt/m^2.*

From Boyle's law $PV=\text{const}=k$

Pressure P of a sphere of radius $R=k/(\frac{4}{3}\pi R^3)=K/R^3$.

Differentiation gives $dP/dR=-3K/R^4=-3P/R$

or $$dR=-(R/3P)dP.$$

The change in pressure due to the charge q is given by

$$-dP=-\sigma^2/2\varepsilon_0=-(q/4\pi R^2)^2/2\varepsilon_0.$$

$$\therefore \quad dR=-\frac{R}{3P}dP=\frac{R}{3P}\cdot\frac{q^2}{32\pi^2\varepsilon_0 R^4}=\frac{q^2}{96\pi^2\varepsilon_0 PR^3}$$

Given $q=20$ stat coul $=20$ e.s.u. of charge

$$=20\times 3.36\times 10^{-10} \text{ coul.}$$

Thus the substitution of numerical values gives

$$dR=5.3\times 10^{-9} \text{ cm.}$$

Example 19. *In Milikan's experiment an oil drop of radius 10^{-4} cm. remains suspended between the plates which are 1 cm apart. If the drop has charge of 5e over it, calculate the potential difference between the plates. The density of oil may be taken as 1.5 gm./cc.*

Since the drop remains suspended between the plates, the viscous force on the drop is zero and the weight mg of the drop is balanced by the force due to electric field between the plates. If V be the potential difference between the plates, then

$$qE=mg \quad \text{or} \quad qV/d=\tfrac{4}{3}\pi r^3(\rho-\sigma)g.$$

In this problem $q=5e=5\times 1.6\times 10^{-19}$ coul, $d=1$ cm$=0.01$ m, $r=10^{-4}$ cm$=10^{-6}$ m, $\rho=1.5\times 10^3$ kg/m^3. Assuming the density of air as negligible in comparison the oil density, we get

$$V=\frac{0.01}{5\times 1.6\times 10^{-19}}\times\frac{4}{3}\times 3.14\times(10^{-6})^3\times 1.5\times 10^3\times 9.81$$

$$=770 \text{ volts.}$$

Oral Questions—

1. Given an insulated rod, how could you determine if it was charged ? How could you find the sign of the charge on it ?
2. The quantum of charge is 1.6×10^{-19} coul, is there any single quantum of mass ?
3. Give some examples of physical quantity to be (*a*) quantized or (*b*) conserved.
4. Whether the constant of proportionality in Coulomb's law is a measured value or a calculated value ?
5. A stone is placed in the gravitational field of earth. Can we also say that the earth lies in the gravitational field of the stone ?

6. Electric lines of force never cross, why ?

7. Two point charges are placed at a certain distance apart. The electric field strength is zero at one point between them. What can you conclude about the charges ?

8. What is the electric flux through a surface encloses an electric dipole ?

9. A charge q is placed inside an enclosure. Compare the total flux coming out of the walls of the enclosure, if it is spherical. cubic, rectangular, parallellopiped or hemisphere.

10. Show that no work is done in moving the test charge from point to point on the surface of the metal.

11. A man is placed with an electrical measuring instrument inside a large closed metal sphere. What will the man observe as (*a*) charge is placed on the sphere, or (*b*) a large charged object is brought close to the sphere ?

12. As you penetrate a uniform sphere of charge, find the effect on **E** while moving towards the centre.

13. A spherical rubber balloon carries a charge that is uniformly distributed over its surface. How does **E** vary for points (*a*) inside, (*b*) on the surface and (*c*) outside the balloon, as the balloon is blown up ?

14. In Milikan's apparatus, how can you find the sign of charge on the droplets from the atomiser.

Problems—

1. In the Bohr model of atomic hydrogen an electron of mass 9.11×10^{-31} kg revolves, about a nucleus consisting one proton, in a circular orbit of radius 5.29×10^{-11} m. If mass of proton is 1.67×10^{-27} kg, calculate the radial acceleration and angular velocity of the electron.

2. If two equally charged balls of identical masses of 0.20 gm are suspended from 50 cm long strings. Calculate the value of each charge, if the strings make an angle of 37° to the vertical. (3.2×10^{-7} *coul*)

3. Two charges Q and Q are placed at the diagonally opposite corners of a square, while charge q and q are placed at the remaining corners. If the resultant force on one charge Q is zero, find the relation between Q and q.

($Q = 2\sqrt{2}q$)

4. A pith ball covered with tin foil having a mass of m kg hangs by a fine silk thread l metre long in an electric field E. When the ball is given electric charge of q coulomb, it stands out d metre from the vertical line. Show that the strength of the electric field is given by

$$E = mgd/q\sqrt{l^2 - d^2} \text{ newton/coulomb.}$$

5. If a uniform sphere of charge q, radius R and charge density ρ has a narrow straight tunnel along its diameter, and a point charge—q' is placed at the entrance to the tunnel and released. Show that its motion is simple harmonic. Find the period of this motion and the maximum velocity acquired by the charge$-q$. Given the mass of the charge q' as m.

$2\pi[3\varepsilon_0 m/q'\rho]^{1/2}$, $R[q'\rho/3\varepsilon_0 m]^{1/2}$.

6. A thin circular ring of radius 20 cm is charged with a uniform charge density of λ coul/m. A small section of 1 cm length is removed from the ring. Find the electric field intensity at the centre of the ring. ($2.25\lambda \times 10^9$ *N/C*)

7. Prove that the electric field intensity due to a uniformly charged ring is maximum at a distance $1/\sqrt{2}$ times its radius form the centre on its axis.

8. The inner surface of a non-conducting hemi-spherical bowl of radius a has uniformly spread charge of surface density σ over it. Find the electric field at the centre of the flat surface of the bowl. ($\sigma/4\varepsilon_0$ N/C)

9. A small sphere whose mass is 1×10^{-4} kg carries a charge of 3×10^{-10} coulomb and is attached to one end of a silk fibre 5 cm long. The other end of the fibre is attached to a large vertical conducting plate which has a surface charge of 25×10^{-6} C/m^2. Find the angle which the fiber makes with the vertical. (41^0)

10. Five thousand lines of force enter a certain volume of space and three thousand lines emerge from it. What is the total charge in coulombs within the volume? (-1.77×10^{-8} *coulomb*)

11. A spherical shell of outer and inner radii a and b respectively is concentric to a metal sphere (of radius c) inside the shell. Find the net charge on the outer surface of the spherical shell, if the electric field at a point P outside the shell at a distance 40 cm from the centre is $200N/C$. (3.55×10^{-9} *coulomb*)

12. A cylinder of radius b is uniformly charged with a volume charge density ρ coul/m^3. Find expressions for the electric field as a function of r for inside and outside the cylinder, if the charge density varies as $\rho=\rho_0 r^3$ within the cylinder. ($\rho_0 r^4/5\varepsilon_0$, $\rho_0 b^5/5\varepsilon_0 r$)

13. Calculate the electric field intensity at a point (i) inside and (ii) outside the spherically symmetric charge distribution of radius a. The charge volume density varies as $\rho(r)=\rho_0\,(1-r^3/a^3)^{1/2}$ inside and is zero outside the distribution.

$$\frac{2a^3\rho_0}{9\epsilon_0 r^2}\left[\;1-\left(1-\frac{r^3}{a^3}\right)^{3/2}\right],\;\frac{2a^3\rho_0}{9\varepsilon_0 r^2}.$$

14. A circular ring of radius a carries a charge which varies as $\lambda=\lambda_0\sin\theta$. Find the electric field at the centre of the ring. ($\lambda_0/4a\varepsilon_0$)

15. A long conducting cylinder of length l carrying a total charge $+q$ is surrounded by a conducting cylindrical shell of total charge $-2q$. Calculate **E** (i) at point outside the shell and (ii) in the region between the cylinders.

($q/2\pi\varepsilon_0 lr$ *radially inward.* $q/2\pi\varepsilon_0 lr$ *radially outward*)

16. Three infinite metal plates A, B, C are arranged parallel to each other. Plate B carries a uniform +ve charge having a charge density σ on both the sides. Plate A and C carry unknown charges, but the arrangement produces zero electric field in the region left to plate A and right to plate C, *i.e.*, outside this arrangement. Find (i) the electric intensity at points between A and B, (ii) charges on the two surfaces of plate A. ($\sigma_{right}=\sigma$, $\sigma_{left}=0$)

17. Two parallel infinite wires carry uniform charges of λ_1 and λ_2 coul/m. If the separation of the wires is b, find the force on unit length of one as a result of the other. ($\lambda_1\lambda_2/2\pi\varepsilon_0 b$)

18. Two concentric thin metallic spherical shells of radii R_1 and R_2 where $R_1 < R_2$ bear charges q_1 and q_2 coulombs respectively. Using Gauss Theorem, show that

(a) The electric field intensity at radius $r < R_1$ is zero.

(b) The electric field intensity at radius r between R_1 and R_2 is $q_1/4\pi\varepsilon_0 r^2$

and (*c*) the electric field intensity at radius $r < R_2$ is $(q_1+q_2)/4\pi\epsilon_0 r^2$.

19. An infinitely long wire is stretched horizontally 4 meters above the surface of the earth. It has a charge of microcoulomb per cm of its length. Calculate the electric field at a point on earth vertically below the wire.

(4.5×10^5 *nt/coul*)

20. The surface density of charge on a conductor, in dry air, is 2.67×10^{-13} coul cm^{-2}. What is the force per unit area due to this charge.

(4.03×10^{-7} *newton/m*^{2})

21. Find the greatest charge which can be carried by a metallic sphere o 10 cm diameter, if the dielectric strength of air is 2×10^4 volts/cm.

(0.55 *micro coulomb*)

22. An α-particle, approaching the surface of a gold nucleus, is at a distance equal to the nuclear radius (6.9×10^{-15} m) from the surface. If the α-particle is assumed as point charge and of mass 6.7×10^{-27} kg, calculate the force and acceleration of the α-particle. (1.9×10^2 *N*, 2.8×10^{28} *m/sec*^{2})

23. A 10 MeV α-particle moves head on towards a stationary nucleus having 80 protons. Caleulate the distance of its closest approach.

(1.44×10^{-15} *m*)

24. Electric charge of uniform density ρ per unit volume is distributed within a spherical shell of inner and outer radii '*a*' and '*b*' respectively. Calculate the electric field at a point distant *r* from the centre such that (*i*) $a<r<b$ and (*ii*) $r<b$

$$\frac{\rho}{3\varepsilon_0}\left[\frac{r^3-a^3}{r^2}\right], \; \frac{\rho}{3\epsilon_0}\left[\frac{b^3-a^3}{r^2}\right]$$

25. An oil droplet of mass 3×10^{-11} gm and of radius 2×10^{-4} cm carries 10 excess electrons. What is its terminal velocity (*a*) when falling in a region in which there is no electric field, (*b*) when falling in an electric field 3×10^5 *N/C*, directed downward? Given the viscousity of air$=1.80 \times 10^{-7}$ *N* sec./m^2.

(4.34×10^{-4} *m/sec.*, 2.74×10^{-4} *m/sec. upward*)

26. A bubble of radius *a* is formed from a soap solution having a surface tension *T*. If the bubble is raised to a potential *V* by being touched with a wire from a static machine, show that the radius of the bubble increases to *r*, given by

$$P(r^3-a^3)+4T(r^2-a^2)-\tfrac{1}{2}\varepsilon_0 V^2 r=0,$$

where *P* is the atmospheric pressure.

3

Electrical Potential

3.1. LINE INTEGRAL OF ELECTRIC INTENSITY

In mechanics, the line integral plays a very fundamental role in problems involving work. Its importance is equally great in electrostatic problems. We know that if a force **F** acts on a body while the body suffers a displacement $d\mathbf{l}$, the work done

$$dW=\mathbf{F}\cdot d\mathbf{l}=F\cos\theta\, dl, \qquad \ldots(1)$$

where θ is the angle between **F** and $d\mathbf{l}$ and $F\cos\theta$ is the component of **F** along the direction of displacement. For a path of finite length, work

$$W=\int_A^B \mathbf{F}\cdot d\mathbf{l}, \qquad \ldots(2)$$

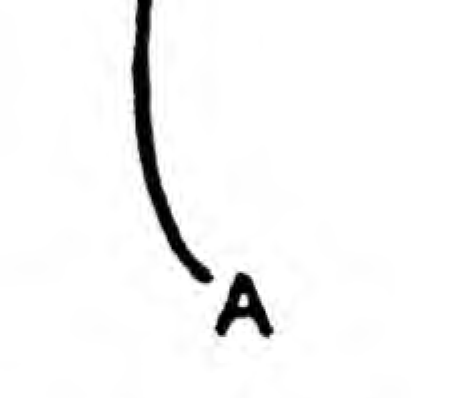


Fig. 3.1.

It is the another use of scalar product of two vectors, one was discussed in chapter 2 where it was applied to the surface integral. One should not confuse as the *integral for flux is taken over a surface and the integral for work is taken along a line.*

We know that the earth's gravitational field, a vector field, is a *conservative field. For a conservative field the potential energy of a system is independent of the path taken and is defined by its position only*, or in other words the *work necessary to move a body from one point to another is independent of the path taken between the two points.* If we can prove that the electrostatic field, *an electric field due to stationary charges,* is a conservative field, then we can say that the use of work and energy is as natural in *electrostatics* as in *mechanics.*

To show that the electrostatic field is a conservative field, let us consider the field due to a point charge q (which is very small so that its presence does not affect the electric field in which it is being placed). If **E** is the electric field at any point P due to the point charge q, the force on the test charge q_0 will be $q_0\mathbf{E}$. Hence the external force $\mathbf{F}=-q_0\mathbf{E}$ is required to displace the test charge q_0. External work necessary to move the test charge through a